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A family of improved Falkner-type methods for oscillatory systems

Jiyong Li [a,b,*]

[a] College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang 050024, PR China
[b] Hebei Key Laboratory of Computational Mathematics and Applications, Hebei Normal University, Shijiazhuang 050024, PR China

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ABSTRACT

For solving general second-order initial value problems $u''(t) = f(t, u(t), u'(t))$, the classical Falkner methods can date back to Falkner's work in 1936. In this paper, we propose and study a family of improved Falkner-type methods for the oscillatory system $u''(t) + Mu(t) = g(t, u(t))$, where g: $R \times R^d \to R^d$, in which the first derivative does not appear explicitly, and $M \in R^{d \times d}$ is a symmetric positive semi-definite matrix. The new methods take into account the oscillatory structures of the problem and exactly integrate the unperturbed problem $u''(t) + Mu(t) = \mathbf{0}$. The global error bounds of the new methods are presented. Numerical experiments are performed to show that the new methods are more efficient than other effective methods appeared in the scientific literature.

1. Introduction

This paper focuses on the system of oscillatory second-order differential equations

$$\begin{cases} u''(t) + Mu(t) = g(t, u(t)), & t \in [t_0, T], \\ u(t_0) = u_0, & u'(t_0) = u'_0, \end{cases} \tag{1}$$

where g: $R \times R^d \to R^d$ and $M \in R^{d \times d}$ is a symmetric positive semi-definite matrix that implicitly contains the frequencies of the system. Such a system is of huge importance in mechanics, quantum physics and electronics, and so on, and it is known that there are two different approaches to dealing with the system: general-purpose numerical methods and adapted methods. However, in essence, the adapted integrators are more efficient since they take advantage of the special structure transpired from the oscillatory system (see [1–8], for example).

For the general second-order initial value problem

$$\begin{cases} u''(t) = f(t, u(t), u'(t)), & t \in [t_0, T], \\ u(t_0) = u_0, & u'(t_0) = u'_0, \end{cases} \tag{2}$$

one of the effective integration methods is due to Falkner [9]. The explicit version can be written as

$$u_{n+1} = u_n + hu'_n + h^2 \sum_{j=0}^{k-1} \beta_j \nabla^j f_n, \quad u'_{n+1} = u'_n + h \sum_{j=0}^{k-1} \gamma_j \nabla^j f_n, \tag{3}$$

* Corresponding author at: College of Mathematics and Information Science, Hebei Normal University, Shijiazhuang 050024, PR China.
E-mail address: ljyong406@163.com

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where $f_n = f(t_n, u_n, u'_n)$ and $\nabla^j f_n$ denotes the jth backward difference. Likewise, the implicit formulas [10] are given by

$$u_{n+1} = u_n + hu'_n + h^2 \sum_{j=0}^{k} \beta_j^* \nabla^j f_{n+1}, \quad u'_{n+1} = u'_n + h \sum_{j=0}^{k} \gamma_j^* \nabla^j f_{n+1}. \tag{4}$$

The coefficients β_j, γ_j, β_j^* and γ_j^* are determined by the generating functions (see [9,10]). For the variable step version of Falkner methods, the reader is referred to [11].

Recently, for the special second-order initial value problem

$$\begin{cases} u''(t) = f(t, u(t)), & t \in [t_0, T], \\ u(t_0) = u_0, & u'(t_0) = u'_0, \end{cases} \tag{5}$$

the reformed Falkner-type methods are proposed by Ramos and Lorenzo [12]:

$$u_{n+1} = u_n + hu'_n + h^2 \sum_{j=0}^{k-1} \beta_j \nabla^j f_n, \quad u'_{n+1} = u'_n + h \sum_{j=0}^{k} \gamma_j^* \nabla^j f_{n+1}, \tag{6}$$

in which u_{n+1} and u'_{n+1} are evaluated by using the first formula of (3) and the second formula of (4), respectively. Because the absence of the first derivative in the right-hand side function f, the value f_{n+1} can be calculated from u_{n+1} straightforwardly. This makes the methods (6)explicit and then the methods can be implemented at low cost.

It is noted that the authors in [12] show that the convergence order of a k-step explicit scheme (6) is $k+1$, but in contrast it is known that a classical k-step explicit Falkner method (3) has only convergence of order k.

The purpose of this paper is to propose a new improvement of k-step explicit scheme (6) which is adapted to the oscillatory problem (1). The new methods (which will be called *improved Falkner-type methods*) can exactly integrate the unperturbed problem $u''(t) + Mu(t) = \mathbf{0}$, but the reformed Falkner-type methods (6) cannot. Moreover, we show that the convergence order of the new improved k-step method is $k+1$.

The rest of this paper is organized as follows: In Section 2, we formulate the new family of improved Falkner-type methods. In Section 3, we consider global error analysis for the new methods and present the error bounds. Section 4 is devoted to the stability analysis for our new methods. Numerical examples are included in Section 5. We conclude the paper with some comments in Section 6.

2. The formulation of improved Falkner-type methods

In order to formulate the improved Falkner-type methods, we restate the definition of the matrix-valued ϕ-functions:

$$\phi_0(M) = \sum_{k=0}^{\infty} \frac{(-1)^k M^k}{(2k)!}, \quad \phi_1(M) = \sum_{k=0}^{\infty} \frac{(-1)^k M^k}{(2k+1)!}, \quad \forall M \in R^{d\times d}, \tag{7}$$

which can be found in [13].

Set $V = h^2 M$. Then it is known that the true solution of (1) and its derivative satisfy the following variation of constants formula [14]

$$\begin{aligned} u(t_n + h) &= \phi_0(V)u(t_n) + h\phi_1(V)u'(t_n) + h^2 \int_0^1 (1-z)\phi_1((1-z)^2 V)g(t_n + hz, u(t_n + hz))dz, \\ u'(t_n + h) &= -hM\phi_1(V)u(t_n) + \phi_0(V)u'(t_n) + h\int_0^1 \phi_0((1-z)^2 V)g(t_n + hz, u(t_n + hz))dz, \end{aligned} \tag{8}$$

where the integral of a matrix-valued function or a vector-valued function is understood as componentwise.

In [8], the authors constructed the adapted Falkner-type methods by replacing $g(t_n + hz, u(t_n + hz))$ in the first equation of (8) and $g(t_n + hz, u(t_n + hz))$ in the second equation of (8) with

$$p_n(t_n + \theta h) = \sum_{j=0}^{k-1} (-1)^j \binom{-\theta}{j} \nabla^j g_n,$$

and

$$p_n^*(t_n + \theta h) = \sum_{j=0}^{k} (-1)^j \binom{-\theta + 1}{j} \nabla^j g_{n+1},$$

respectively, where $g_j = g(t_j, u_j)$, $u_j \approx u(t_j)$ and $\nabla^j g_n$ denotes the jth backward difference. The adapted Falkner-type methods proposed in [8] exactly integrate the unperturbed problems $u''(t) + Mu(t) = \mathbf{0}$ and when $M \to \mathbf{0}(V \to \mathbf{0})$, they reduce to the reformed Falkner methods (6).

In this paper, we will formulate a new family of improved Falkner-type methods in a different way from [8]. The new k-step improved Falkner-type methods are shown to have the same convergence order as the k-step adapted Falkner-type