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# Analytical solutions for heat diffusion beyond Fourier law

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ABSTRACT

We obtain solutions for differential equations, describing a broad range of physical problems by the operational method with recourse to inverse differential operators, integral transforms and operational exponent. Generalized families of orthogonal polynomials and special functions are also employed with recourse to their operational definitions. The evolutional type problems for heat transfer in various heat conduction models are studied. Exact analytical solutions for Guyer–Krumhansl hyperbolic heat equation are obtained and compared with those of Fourier and Cattaneo equations. Modelling heat pulse propagation from a laser source is performed in the framework of Fourier, Cattaneo and Guyer–Krumhansl heat transfer models. Compliance of obtained solutions with the maximum principle is studied.



## 1. Introduction

Differential equations (DE) play very important role in mathematics. Moreover, they are of great importance also in physics since they describe many physical processes. Sometimes, the same DE describes more than one physical process in different branches of physics. Thus, obtaining the solutions for differential equations is of paramount importance. Few types of differential equations have explicit analytical solutions. During the last decade the numerical calculations and computer methods have advanced greatly. They facilitate equations solving. Numerous numerical methods for solving differential equations exist (see, for example, [1–6]. However, understanding of the obtained solutions and of the interplay of various parameters in them can be best done in analytical form. To this end expansion in series of generalized forms of orthogonal Hermite, Laguerre and others polynomials are employed (see, for example, [7–11]). Operational approach allows easy solving of a broad class of differential equations and relevant physical problems. It has been successfully applied for treatment of the problems, related to propagation and radiation of accelerated charges, undulator radiation (UR) [12–15], heat and mass transfer [16–19]. In these studies generalized forms of special functions and polynomials naturally arise [20–23]. Some examples of solutions of the heat diffusion and other equations by the inverse derivative method were considered in [16,24,25]. The studies of hyperbolic heat conduction phenomena were reviewed in [26,27]. Extensive study is presented in the book [28]. Other important studies were conducted, for example, in [29–32]. In what follows we shall apply operational approach, combined with integral transforms and extended forms of orthogonal polynomials to obtain exact analytical solutions for heat diffusion equations beyond Fourier model.

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## 2. Fourier and non-Fourier heat conduction

The Fourier's law [33] is an explicit scheme of heat conduction modelling within continuum physics as it provides an excellent agreement between theory and experiment for more than 90% of heat propagation cases. It relates linearly the cause—the temperature gradient, to the effect—the heat flux:

$$\partial_t T = \alpha \nabla^2 T, \tag{1}$$

where $\alpha = k_T$ denotes heat diffusivity. The solution of the above equation with the initial condition $T(x, 0) = f(x)$ is given by the Gauss-type integral:

$$T(x,t) \equiv \hat{S} f(x) = \frac{1}{2\sqrt{\pi \alpha t}} \int_{-\infty}^{\infty} \exp\left\{ -\frac{(x-\xi)^2}{4\alpha t} \right\} f(\xi) d\xi, \tag{2}$$

where the heat operator $\hat{S}$ reads as follows:

$$\hat{S} = \exp(\alpha t \partial_x^2). \tag{3}$$

However, Fourier's law has some contradictions and shortcomings, among which, for example, infinite speed of the heat signal propagation. Indeed, it was noted by L. Onsager in 1931 that the Fourier's model was in contradiction with the principle of microscopic reversibility [34] and this contradiction "is removed when we recognize that is only an approximate description of the process of conduction, neglecting the time needed for acceleration of the heat flow". The Fourier's law is unsuitable for description of many low temperature (<25°K) heat phenomena. Moreover, it has the following unphysical property: if an instant temperature disturbance is applied at a point in the solid, it is felt instantly at any distance in the body. To describe the so-called second sound, when temperature disturbance propagates like damped waves, Cattaneo [35] proposed a time-dependent relaxational model, which yielded in one dimension the following equation:

$$(\tau \partial_t^2 + \partial_t) T = k_T \nabla^2 T, \tag{4}$$

where $k_T$ is the thermal diffusivity and the relaxation time $\tau$ in heat conduction is extremely small ($\tau \approx 10$–$13$ s) at room temperature. Cattaneo's eq. (4) describes heat transfer in the form of wave propagation at finite velocity: the ratio $v_t = \sqrt{k_T/\tau}$ is indeed a velocity like quantity, representing the speed of the heat wave in the medium. The material parameter $\tau$ is an intrinsic thermal property, representing the build-up period for the start of a heat flow after a temperature disturbance was imposed at the boundary of the domain. In Cattaneo's model the heat flow starts with a delay time $\tau$ after the application of the temperature gradient. This relaxation time $\tau$ is associated with the linkage time of phonon–phonon collision, needed for the heat to start flowing, and in a sense, it measures the thermal inertia of the matter. Eq. (4) is the simplest model of the second sound phenomenon observed first in liquid Helium [36]. Later on, the analysis of the theoretical background resulted in the observation of second sound also in solid crystals [37], via properly designed experiments [38–40]. In these tests the heat pulse technology was crucial for the sensitive detection of the thermal diffusivity. In what follows we will demonstrate how to analytically solve Cattaneo's equation and its extended forms by the operational method. However, even Cattaneo's eq. (4) does not eliminate all the problems. Although it leads to a finite value for the wave velocity, the latter is not in agreement with ultrasonic wave propagation in dilute gases and it does not describe heat pulse propagation in non-metallic very pure crystals, like Bi or Na F at very low temperature. So-called ballistic transport should be taken into account; it occurs when the mean free path of a particle significantly exceeds the dimension of the medium in which it travels. For example, for electrons in a medium with negligible electric resistance, their motion is altered by collisions with the walls. The mean free path can be increased by reducing the number of impurities in a crystal or by lowering its temperature. Ballistic conduction is not limited to electrons but can also apply to phonons. The ballistic conditions are typical for very thin films and wires. It is typically observed in such quasi one- or two-dimensional structures as graphene, carbon nanotubes, silicon nanowires etc. (see, for example, [41,42–44,45]). In the ballistic conditions, when the mean free path of a particle significantly exceeds the dimension of the medium in which it travels, neither Casimir phonon [46] nor Fourier diffusion [33] theories are accurate and thermal transfer is influenced by both internal and boundary scattering. Most known further generalization of the Fourier law is based on the Guyer–Krumhansl (GK) equation [47], which in one dimension reduces to the following form:

$$\left(\partial_t^2 + \varepsilon \partial_t - \delta \partial_{t,x,x}^3\right) T(x,t) = \alpha \partial_x^2 T(x,t). \tag{5}$$

Guyer and Krumhansl solved the linearized Boltzmann equation for a phonon field in dielectric crystals at low temperature and derived a non-local extension of Cattaneo's equation. It is well adapted to the description of phonon gases, where heat transport is not only governed by diffusion (like in Fourier's description) and second sound (like in Cattaneo's model), but, in addition, by ballistic transport. Moreover, some recent heat propagation experiments at room temperature on microscopic [48] and on macroscopic samples [49–53] also demonstrate deviation from Fourier and Cattaneo's law. The observed phenomenon was considered typical for the GK relation [54], despite there was no phonon transport there. In what follows we shall address these equations and obtain analytical solutions for them using operational approach. Other extensions and modifications of the GK equation predict more terms and they will be discussed elsewhere.