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Numerical investigation of noise induced changes to the solution behaviour of the discrete FitzHugh–Nagumo equation

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ABSTRACT

In this work we introduce and analyse a stochastic functional equation, which contains both delayed and advanced arguments. This equation results from adding a stochastic term to the discrete FitzHugh–Nagumo equation which arises in mathematical models of nerve conduction. A numerical method is introduced to compute approximate solutions and some numerical experiments are carried out to investigate their dynamical behaviour and compare them with the solutions of the corresponding deterministic equation.

1. Introduction

The main purpose of this paper is to analyse a stochastic mixed-type functional differential equation, which may be written in the form:

$$dy(t) = [A(y(t+\tau) - 2y(t) + y(t-\tau)) + By(t))(y(t)-1)(\alpha - y(t))]dt + \gamma y(t)(y(t)-1)dW(t), \tag{1}$$

where α, A, B, τ are known real parameters. In the next section we will discuss in detail the boundary conditions we consider in the paper. As far as we are aware, the subject of stochastic mixed-type functional differential equations (SMTFDE) is new in the literature and we begin by explaining our motivation to introduce this type of equation. In some recent works [9,17] we were concerned about a deterministic MTFDE:

$$y'(t) = A[y(t+\tau) - 2y(t) + y(t-\tau)] + By(t)(y(t)-1)(\alpha - y(t)), \tag{2}$$

sometimes known as the discrete FitzHugh–Nagumo equation, arising in nerve conduction theory. The history of the analysis of this equation began in 1952, when Hodgkin and Huxley [12] introduced a mathematical model that describes the excitation and flow of electrical current through the surface of a giant nerve fibre from a squid. This investigation was continued in the works of FitzHugh [6,7] and Nagumo [24].

During its evolution the FitzHugh–Nagumo equation has taken different forms, and after various simplifications it appeared in the work [5] in the form (2), where $y(t)$ represents the potential at a Ranvier node of the axon at the moment t (in this case, the potential at the neighbouring nodes is denoted by $y(t-\tau)$ and $y(t+\tau)$); the constant τ is the time that a signal takes to be transmitted from a node to the neighbouring one (in other words, τ is inversely proportional to the propagation speed of the signal). The constants A, B reflect the resistance and the conductance in the nerve axon, while α is the threshold potential.

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From a mathematical point of view, an important feature of Eq. (2) is that it contains both negative and positive deviations of the argument (delayed and advanced terms); this is the reason it is called a mixed type functional differential equation (or an advance-delay-differential equation). Important contributions to the analysis of this type of equation have appeared in the literature in the last two decades of the past century. The ill-posedness of mixed-type functional differential equations was discussed by Rustichini [29], where he considered linear autonomous equations. He extended his results to nonlinear equations [30]. Mallet-Paret applied Fredholm theory to obtain new results for this class of equation [20] and introduced the idea of factorisation of their solutions [21]. More recently, Hupkes and Verduyn Lunel studied the behaviour of solutions to nonlinear autonomous mixed-type functional differential equations in the neighbourhood of an equilibrium solution [13]. Based on existing insights into the qualitative behaviour of MTFDEs, the authors of [8] developed a new approach to the analysis of these equations in the autonomous case. More precisely, they analysed MTFDEs as boundary value problems, that is, for a linear MTFDE they considered the problem of finding a differentiable solution on a certain real interval $[-1, k]$, $k \in \mathbb{N}$, given its values on the intervals $[-1, 0]$ and $(k-1, k]$. This approach was developed further in [31], where new numerical methods were proposed for the solution of such boundary value problems. In [10,16], these methods were extended to the non-autonomous case and new results were obtained about their numerical analysis. Once efficient computational methods had been created for the numerical treatment of linear MTFDEs, the next step was to extend these methods to the case of nonlinear ones, which includes equations of the form (2). This was done by Abell et al. [1] and then by the authors of the present paper in [9,17].

Up to now we have been concerned only with deterministic equations such as (2). However the study of stochastic problems such as (1) is of great importance for the modelling of excitable neural systems. Noise-induced effects on nonlinear systems have attracted great research interest in the last 20 years. In particular scientists have been interested in stochastic resonance, which occurs when a periodic input signal in a nonlinear system can be amplified under the effect of noise. In some cases, noise can play a constructive role in excitable neural systems, even when there is no deterministic input signal; this phenomenon is named coherence resonance. Both stochastic and coherence resonance can be enhanced when a resonator is coupled into a chain of identical ones. In [18] the authors show how noise, coupling and bi-stable dynamics cooperate to organise spacial order, temporal periodicity and peak signal-to-noise ratio. The effect of coherence resonance in a heterogeneous array of coupled FitzHugh–Nagumo neurons is demonstrated in [32]. In both cited works only one-dimensional arrays of excitable neurons are considered, such that each neuron can interact only with two neighbours. This case is modelled by equations similar to (1). The case of two-dimensional arrays has been also considered (see for example [26]). In that paper the author shows that there is an optimal noise intensity, for which the inherent spatial periodicity of the system is resonantly pronounced. In the two-dimensional case the focus is often shifted towards networks with variable random connectivity. In these systems, known as small-world networks, there is a certain fraction of so-called long-range couplings (or shortcut links) that connect distant units of the system, while all other units are coupled in a diffusive-like manner. In [15] the authors investigate the effect of small-world connectivity on the phenomenon of coherence resonance of Hodgkin–Huxley neurons. In [27] it is shown that that the temporal order, characterised by the auto-correlation of a firing-rate function, can be enhanced by small-world connectivity. On the other hand, the introduction of long-range couplings induces disorder in otherwise ordered noise-induced patterns that can be observed in the case of exclusively diffusive connectivity. This latter effect, known as spatial decoherence, is studied in detail in [28].

In our work we aim to compare the dynamical behaviour of the solutions to Eq. (1) with those of (2). A similar comparison has been carried out in [25], in the case of a stochastic delay-differential equation.

This is a very challenging aim, since (as far as we know) there have been no previous attempts at investigating stochastic mixed-type functional differential equations. Therefore, we must rely on insights available from the related field of stochastic delay differential equations.

In the last ten years many works have been devoted to the analysis of different numerical algorithms applied to stochastic delay-differential equations, including the Euler–Maruyama [3,19,22,23], explicit one-step [2], Maruyama multistep [4] and Milstein methods [14].

The outline of this paper is as follows. In Section 2 we recall the most important features of the deterministic equation (2) and introduce numerical methods for its approximation. In Section 3 we introduce the stochastic equation (1) and describe a numerical method for its approximation, based on the algorithm for the deterministic equation. In Section 4 we present and discuss numerical results. We finish with some conclusions in Section 5.

2. Analysis and approximation of the deterministic equation

2.1. Building the forward and backward solutions

We consider the following equation:

$$y'(t) = A(y(t+\tau) - 2y(t) + y(t-\tau)) + Bf(y(t)), \tag{3}$$

where A, B, τ are given constants,

$$f(x) = x(x-1)(\alpha - x), \tag{4}$$