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The numerical solution of nonlinear two-dimensional Volterra–Fredholm integral equations of the second kind based on the radial basis functions approximation with error analysis

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ABSTRACT

In this paper, we present a numerical method for solving two-dimensional nonlinear Volterra–Fredholm integral equations of the second kind. The method approximates the solution by the discrete collocation method based on radial basis functions (RBFs) constructed on a set of disordered data. The proposed method is meshless, since it does not require any background mesh or domain elements. Error analysis of this method is also investigated. Numerical examples which compare the proposed method with 2D-TFs method [4] approve its supremacy in terms of accuracy and computational cost. Using various RBFs we have concluded that MQ-RBF is the best choice for the proposed method.

1. Introduction

Integral equations are used as mathematical models for many varied physical situations, and they also occur as reformulations of other mathematical problems. Nonlinear integral equations have been studied in connection with many diverse topics such as vehicular traffic, biology, the theory of optimal control, economics, etc [12,13].

We consider the two-dimensional Volterra–Fredholm integral equation

$$u(x,t) - \int_0^t \int_0^x k_1(x,t,y,z)\psi_1(y,z,u(y,z))dydz + \int_0^1 \int_0^1 k_2(x,t,y,z)\psi_2(y,z,u(y,z))dydz = f(x,t). \tag{1}$$

The functions $f(x, t)$, $k_1(x, t, y, z)$ and $k_2(x, t, y, z)$ are assumed to be given smooth real valued functions on $\Omega = [0,1] \times [0,1]$ and $D = \{(x,t,y,z) : 0 \le y \le x \le 1, 0 \le z \le t \le 1\}$, respectively and $u(x, t)$ is the solution to be determined. Classical theorems on the existence and uniqueness of the solution of two-dimensional nonlinear integral equations can be found in Abdou et al. [1,2]. Theoretical results and numerical approaches on one dimensional integral equations have been investigated by several authors [14,15] but a few studies have been devoted to two-dimensional integral equations. Han and Wang [16] studied the iterated collocation method to solve two-dimensional nonlinear Volterra integral equations. The Nystrom method was used to solve two-dimensional nonlinear Fredholm integral equations in Han and Wang [17,18]. Recently a numerical method based on the Tau method is presented for solving two-dimensional nonlinear Volterra–Fredholm integral equations [21].

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Table 1
Some well-known functions that generate RBFs.

Name of function	Definition
Gaussian (GA)	$\phi(r) = \exp(-cr^2), \quad c > 0$
Multiquadrics (MQ)	$\phi(r) = (r^2 + c^2)^{\frac{1}{2}}$
Inverse multiquadrics (IMQ)	$\phi(r) = (r^2 + c^2)^{\frac{-1}{2}}$
Thin plate splines	$\phi(r) = (-1)^{k+1} r^{2k} \log(r), \quad k \in \mathbb{N}$

In recent years, meshless methods have been used in many different areas ranging from artificial intelligence, image processing, neural networks, and sampling theory, etc. Radial basis functions (RBFs) are probably best known for their applications to problems with scattered data. The multiquadric (MQ) RBF interpolation method was developed by Roland Hardy who described and named the method in a paper appeared in 1971 [6]. Among the tested RBFs, the MQ-RBFs gave the most accurate results. In 1990 Kansa first used the MQ-RBFs method to solve differential equations [7,8]. Kansa's method was recently extended to solve various ordinary and partial differential equations [9–11]. We refer interested reader to Refs. [22–24] for applications of meshless methods for finding numerical solution of integral equations.

The main purpose of this paper is to present a numerical method based on radial basis functions approximation for numerical solution of nonlinear two-dimensional Volterra–Fredholm integral equations. The outline of this paper is as follows: In Section 2 we review some basic formulations and properties of the radial basis functions approximations. In Section 3, we present a computational method for nonlinear two-dimensional Volterra–Fredholm integral equations utilizing the RBF approximation. We provide the error analysis for the method in Section 4. Finally, numerical examples are given in Section 5.

2. Radial basis functions approximation

A meshfree method does not require a mesh to discretize the domain of the problem under consideration, and the approximate solution is constructed entirely based on a set of scattered nodes. Radial basis functions is one of the most developed meshless methods that has attracted attention in recent years and form a primary tool for multivariate interpolation [11]. It is also receiving increasing attention for solving PDE's on irregular domains. We quote the following definitions from Ref. [3].

Definition 2.1. A function $\Phi : \mathbb{R}^d \to \mathbb{R}$ is said to be radial if there exists a function $\phi : [0, \infty) \to \mathbb{R}$ such that $\Phi(\mathbf{x}) = \phi(\|\mathbf{x}\|_2)$, for all $\mathbf{x} \in \mathbb{R}^d$.

Some of the most popular RBFs are given in Table 1. In order to explain the multivariate scattered data interpolation by radial basis functions, consider the following definition:

Definition 2.2. A real-valued continuous even function Φ is conditionally positive definite of order m, if for all sets $X = \{x_1, \ldots x_N\} \subset \mathbb{R}^d$ of distinct points, and all vectors $\mathbf{\Lambda} = [\lambda_1, \ldots, \lambda_N]^T \in \mathbb{R}^N$ satisfying $\sum_{i=1}^{N} \lambda_i p(x_i) = 0$ for all polynomials of degree less then m the quadratic form

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \lambda_i \lambda_j \Phi(x_i - x_j), \tag{2}$$

is positive.

In interpolation of the scattered data using RBFs, the approximation of a function f on a set $X = \{x_1, \ldots, x_N\}$ usually has the form:

$$\mathcal{P}f = \sum_{i=1}^{N} \lambda_i \phi(\| x - x_i \|) + \sum_{l=1}^{M} d_l p_l(x), \quad x \in \Omega, \tag{3}$$

where p_1, ..., p_M form a basis for the $M = \binom{d+m-1}{m-1}$ dimensional linear space Π_{m-1}^d of polynomials of total degree less than or equal to $m-1$ in d variables. The interpolation problem is to find $\lambda_i, \quad i = 1, \ldots, N$ and $d_l, \quad l = 1, \ldots, M$ such that the interpolant $\mathcal{P}f$ through all data satisfies

$$\mathcal{P}f(x_i) = f(x_i), \quad i = 1, \ldots, N, \qquad \text{and} \qquad \sum_{i=1}^{N} \lambda_i p_j(x_i) = 0, \, j = 1, \ldots, M. \tag{4}$$

One can write this system in matrix form as

$$\begin{pmatrix} A & P \\ P^T & 0 \end{pmatrix} \begin{pmatrix} \mathbf{\Lambda} \\ \mathbf{d} \end{pmatrix} = \begin{pmatrix} \mathbf{f} \\ 0 \end{pmatrix}, \tag{5}$$

where A and P are the $N \times N$ and $N \times M$ matrices with the elements $A_{ij} = \phi(\|x_i - x_j\|), i, j = 1, \ldots, N$ and $P_{ij} = p_j(x_i), i = 1, \ldots, N, j = 1, \ldots, M,$ respectively. Furthermore $\Lambda = [\lambda_1, \ldots, \lambda_N]^T, \mathbf{d} = [d_1, \ldots, d_M]^T$ and $\mathbf{f} = [f(x_1), \ldots, f(x_N)]^T$.