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Constructions of ϵ-mono-components and mathematical analysis on signal decomposition algorithm

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ABSTRACT

The concept of mono-component is widely used in non-stationary signal processing and time-frequency analysis. A special class of mono-components, called ϵ-mono-components, were proposed in our recent publication. It was illustrated that this model coincides with the intuition of human beings on the physical mono-components very well provided that the parameter ϵ is sufficiently small. It is then very meaningful to construct desired ϵ-mono-components and design algorithms to decompose and represent non-stationary signals adaptively. This paper studies the constructions of ϵ-mono-components and makes mathematical analysis on an adaptive signal decomposition algorithm based on ϵ-mono-components.

1. Introduction

The Fourier transform is a powerful tool for stationary signal processing. By taking the Fourier transform, a signal is decomposed into a band of sinusoidal waves of different frequencies unchanged over time. However, for the non-stationary signals, the frequencies of the signals do vary over time. To study the instantaneous oscillating behavior of a non-stationary signal, the concept of instantaneous frequency (IF) was introduced and plays a key role in non-stationary signal processing and time-frequency analysis. In 1946, Gabor proposed the definition of IF by means of the Hilbert transform [6]. Suppose $x(t)$ is a real-valued 2π-periodic function. To define its IF, one first constructs the analytic signal $s(t) := x(t) + iHx(t)$ in which H is the Hilbert transform defined by the Cauchy principal value of the singular integral

$$Hx(t) := \text{p.v.}\frac{1}{2\pi}\int_0^{2\pi} x(t-s)\cot\frac{s}{2}\,ds. \tag{1.1}$$

Rewriting the analytic signal in the polar form $s(t) = \rho(t)e^{i\theta(t)}$, we call $\rho(t)$ and $\theta(t)$ the analytic amplitude and phase of $x(t)$ (or $s(t)$) respectively. The phase derivative $\theta'(t)$, if exists, is consequently called the IF of $x(t)$ (or $s(t)$).

As discussed in our recent publication [12], the above concept of IF makes sense for physical monocomponent signals only. That is, the signal should contain only one frequency or a narrow range of frequencies at any time [1,2,4,8,13,16]. It is a fundamental question and remains open in the past several decades to establish a rigorous mathematical model for mono-component signals. In 1998, an empirical and practical model for monocomponent signals was proposed, which is called the

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intrinsic mode function (IMF) [9,14,30]. The basic requirement of an IMF is that the upper and lower envelopes defined by the cubic spline interpolation are symmetrical with respect to the time axis. But strictly speaking, this symmetrical condition is nearly impossible to be met since the upper and lower envelopes are obtained through different interpolation points respectively. Moreover, the upper/lower envelopes cannot be regarded as the amplitudes of a signal due to the unavoidable undershoots [9,31] caused by the spline interpolation. In addition, it has been shown that an IMF may contain negative IF [25].

As the advent of the EMD (empirical mode decomposition) at the end of the last century [9,14], the study on mono-component signals has become a hot spot of research. Based on the basic assumption that a monocomponent signal has nonnegative IF, Qian et al. introduced the following concept and termed as 'mono-component' in [17–19,22].

Definition 1. Let $s(t)=\rho(t)e^{i\theta(t)}$ be an analytic signal, where $\rho(t)$ and $\theta(t)$ are respectively the analytic amplitude and phase. If $\theta(t)$ is differentiable and $\theta'(t)\geq 0$, then $s(t)$ (or: its real part $x(t)=\rho(t)\cos\theta(t)$) is called a mono-component and denoted by $s\in\mathcal{M}$ (or: $x\in\mathcal{M}$).

The condition $\theta'(t)\geq 0$ in Definition 1 is presented to guarantee the nonnegativity of the IF since the frequency, which is a quantity to represent the velocity of oscillation of a signal, is nonnegative from the physical point of view. It has been shown that an analytic signal is the nontangential boundary value of a holomorphic function in the Hardy space [7,17]. Therefore, mono-components can be studied through the classical theory of the Hardy space. Conditions for $\rho(t)e^{i\theta(t)}$ to be an analytic signal or a mono-component were studied deeply in [10,17,19,26,29,32]. Several important families of mono-components with nonlinear phases were constructed based on the Blaschke products and the factorization theorem for functions in the Hardy space [17–19,22,28].

The amplitude of a mono-component may not always coincide with its physical amplitude which is the maximum displacement or distance moved by a point on a vibrating body or wave measured from its equilibrium position [5,12]. By analyzing and observing some typical mono-components, a special class of mono-components, called ϵ-mono-components, were presented in our recent publication [12], in which the parameter ϵ is employed to measure the consistency between the analytic amplitude and the physical one. Before the definition of ϵ-mono-components, we introduce two function spaces as follows [12]:

$$C_{\rightarrow}(\mathbb{R}):=\{\phi\in C(\mathbb{R}):\phi \text{ is increasing and } \lim_{t\to\pm\infty}\phi(t)=\pm\infty\},$$
$$C_{\uparrow}(\mathbb{R}):=\{\theta\in C(\mathbb{R}):\theta \text{ is strictly increasing and } \theta(t+2\pi)-\theta(t)\in 2\pi\mathbb{Z},\ \forall t\in\mathbb{R}\},$$

where $\mathbb{R}$ denotes the set of all the real numbers and $\mathbb{Z}$ the set of all the integers, and $C(\mathbb{R})$ denotes the set of all the continuous functions on $\mathbb{R}$. It holds obviously that $C_{\uparrow}(\mathbb{R})\subset C_{\rightarrow}(\mathbb{R})$. Below is the definition of ϵ-mono-components [12]:

Definition 2. Given $\epsilon>0$, a signal $s(t)=\rho(t)e^{i\theta(t)}\in\mathcal{M}$ is said to be an ϵ-mono-component and denoted by $s\in\mathcal{M}_\epsilon$ if $\theta\in C_{\uparrow}(\mathbb{R}),\ \rho(t)\geq 0,\ \forall t\in\mathbb{R}$, and $\rho(t)$ can be expressed as

$$\rho(t)=\lambda_0+\sum_{j=1}^{N}\lambda_j\cos\phi_j(t)$$

for some $\lambda_0,\lambda_j\in\mathbb{R},\ \phi_j\in C_{\rightarrow}(\mathbb{R}),\ j=1,\dots,N,$ satisfying

$$\phi_j'(t)\leq\epsilon\theta'(t),\qquad \text{a. e. } t\in\mathbb{R},\qquad j=1,2,\dots,N. \tag{1.2}$$

The condition $s(t)=\rho(t)e^{i\theta(t)}\in\mathcal{M}$ in Definition 2 guarantees $\rho(t)$ and $\theta(t)$ are respectively the analytic amplitude and phase of the signal, and the IF of the signal has nonnegative values. The condition (1.2) guarantees that the amplitude oscillates much more slowly than the phase part at any time provided that the parameter ϵ is sufficiently small.

In this paper, we first study the constructions of ϵ-mono-components. Then, two mathematical issues, the convergence property and the solution of the minimization problem of the adaptive signal decomposition algorithm presented in [12], are studied. The rest of this paper is organized as follows. The constructions of ϵ-mono-components for a special class of phases and for a given amplitude are studied respectively in Sections 2 and 3. In Section 4, we make mathematical analysis on the adaptive signal decomposition algorithm presented in [12], numerical experiments are also given in this section. Discussion of ϵ-mono-components on the real line is given in Section 5. Finally, Section 6 is the conclusion of the paper.

2. Construction of ϵ-mono-components for a special class of phases

There is a class of fundamental unimodular mono-components called nonlinear Fourier atoms [11,17,18]. A nonlinear Fourier atom $e^{i\theta_a(t)}$ is defined through the boundary value of the Möbius transform, that is,

$$e^{i\theta_a(t)}:=\tau_a(e^{it})=\frac{e^{it}-a}{1-\bar{a}e^{it}},\quad t\in[0,2\pi),\quad 0\leq|a|<1. \tag{2.1}$$

In the following we construct ϵ-mono-component $\rho(t)e^{i\theta(t)}$ with the phase $\theta(t)=n\theta_a(t)$.