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WENO scheme with new smoothness indicator for Hamilton–Jacobi equation

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ABSTRACT

In this paper, we develop a new weighted essentially non-oscillatory (WENO) scheme for Hamilton–Jacobi (HJ) equation by proposing a new family of smoothness indicators, which includes the smoothness indicator in Jiang and Peng (2000) as one of its members. The new family of smoothness indicators has three parameters. By choosing the parameters properly, the new WENO scheme provides more accurate numerical solution than the original one, and increases little computational cost.

1. Introduction

In this paper, we research the fifth order finite difference WENO scheme for the HJ equation

$$\phi_t + H(x, y, t, \phi, \phi_x, \phi_y) = 0, \quad \phi(x, y, 0) = \phi_0(x, y), \tag{1}$$

which appears in many applications, such as differential games, geometric optics and image processing. As time evolves, the HJ equation possibly develops the discontinuous derivative even the initial condition is smooth enough, however its solution is still continuous, thus it is easier to be solved than conservation laws.

For HJ equation, a good high-order numerical scheme should achieve its optimal order of accuracy in smooth region of derivative, and avoid the oscillations near discontinuities of derivative. In order to satisfy above requirements, Jiang and Peng [7] developed a fifth order finite difference WENO scheme on uniform mesh to solve the one-dimensional (1D) HJ equation, and extended it to 2D one in a dimension by dimension fashion. In [7], the derivative ϕ_x is a convex combination of $\phi_x^s (s = 0, 1, 2)$ which are defined on three substencils, and the corresponding nonlinear weights rely on the smoothness indicators which depend on the second and third derivatives of function ϕ. Then Zhang and Shu [16] developed a WENO scheme on the triangular mesh, which is more suitable to deal with the irregular boundary than the uniform mesh. By using a strategy to choose diversified smaller stencils to make up the bigger stencil, they obtained the numerical solution with high order of accuracy and good resolution. After that, the HJ equation was solved by Hermite WENO scheme [13,17], which uses both the function and its first derivative in the WENO reconstruction, thus possesses the major advantage of compactness. The Hermite WENO scheme on uniform mesh was developed in [13], the one on triangular mesh was developed in [17]. The HJ equation was also solved by the central WENO scheme [10], which develops as a Godunov-type scheme, thus its global reconstruction is evolved exactly according to the equation.

In the WENO reconstruction, the nonlinear weight plays an important role, it depends on the smoothness indicator which measures the smoothness of function on each substencil. Liu et al. [11] proposed a smoothness indicator by using

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the undivided difference, then Jiang and Shu [8] developed a more efficient one which greatly improves the WENO scheme. The WENO method is popular for solving the conservation laws and Hamilton–Jacobi equation [1,2]. Note that, in smooth region, if the nonlinear weight approximates the linear weight with higher order truncation error, the corresponding WENO scheme will be closer to the optimal scheme, thus provide more accurate numerical solution. In [7], the nonlinear weight approximates the linear weight only with $O(\triangle x^2)$, where $\triangle x$ is the spatial size. In this paper, we propose a new family of smoothness indicators, which has three parameters, namely $A_l(l=1,2,3)$. When $(A_1,A_2,A_3)=(0,1,1)$, we obtain the original smoothness indicator in [7]. By our research, we find that: (1) by choosing $A_1\neq 0$ for the new family of smoothness indicators, its corresponding nonlinear weight can approximate the linear weight with $\triangle x^4$ in smooth region; (2) among the new family of smoothness indicators, the new smoothness indicator with $(A_1,A_2,A_3)=(1/10,1,1)$ gives more accurate numerical solution than the original one, and increases little computational cost.

The remainder of this paper proceeds as follows. In Section 2, a brief review of the finite difference WENO scheme in [7] is given. In Section 3, a new smoothness indicator is proposed. In Section 4, the performance of finite difference WENO scheme with new smoothness indicator is shown by numerical examples.

2. The fifth-order WENO scheme

2.1. WENO spatial discretization

In this section, we review the fifth order finite difference WENO scheme proposed by Jiang and Peng [7]. Let x_i be a discretization of R^1 with uniform spacing $\triangle x$. We solve the 1D HJ equation $\phi_t+H(x,t,\phi,\phi_x)=0$ by using following semidiscrete form:

$$\frac{d\phi_i}{dt}=L(\phi)_i=-\hat{H}(x_i,t,\phi_i,\phi^+_{x,i},\phi^-_{x,i}), \tag{2}$$

where $\hat{H}$ is a Lipschitz continuous monotone flux which is consistent with H. In this paper we use the global Lax–Friedrichs (GLF) flux as follow

$$\hat{H}^{GLF}(\phi^+_{x,i},\phi^-_{x,i})=H\left(\frac{\phi^+_{x,i}+\phi^-_{x,i}}{2}\right)-\alpha\frac{\phi^+_{x,i}-\phi^-_{x,i}}{2}. \tag{3}$$

In the above, $\alpha=\max_{\phi_x\in\{\phi^+_{x,i}\}\bigcup\{\phi^-_{x,i}\}}|H_1(\phi_x)|$, where H_1 stands for the partial derivative of H with respect to ϕ_x; $\phi^+_{x,i}$ and $\phi^-_{x,i}$ are the numerical approximations to $\phi_x(x_i)$ from right and left respectively. We only give the reconstruction of $\phi^-_{x,i}$ here, the one of $\phi^+_{x,i}$ is similar. Giving a large left-biased stencil $\{x_k, k=i-3,\dots,i+2\}$, we can have three third-order approximations of $\phi_x(x_i)$ based on three substencils $\{x_k, k=i+s-3,\dots,i+s\}(s=0,1,2)$, that is:

$$\begin{cases}\phi^{-,0}_{x,i}=\dfrac{1}{\triangle x}\left(-\dfrac{1}{3}\phi_{i-3}+\dfrac{3}{2}\phi_{i-2}-3\phi_{i-1}+\dfrac{11}{6}\phi_i\right),\\ \phi^{-,1}_{x,i}=\dfrac{1}{\triangle x}\left(\dfrac{1}{6}\phi_{i-2}-\phi_{i-1}+\dfrac{1}{2}\phi_i+\dfrac{1}{3}\phi_{i+1}\right),\\ \phi^{-,2}_{x,i}=\dfrac{1}{\triangle x}\left(-\dfrac{1}{3}\phi_{i-1}-\dfrac{1}{2}\phi_i+\phi_{i+1}-\dfrac{1}{6}\phi_{i+2}\right),\end{cases} \tag{4}$$

then $\phi^-_{x,i}$ is a convex combination of $\phi^{-,s}_{x,i}(s=0,1,2)$:

$$\phi^-_{x,i}=\omega_0\phi^{-,0}_{x,i}+\omega_1\phi^{-,1}_{x,i}+\omega_2\phi^{-,2}_{x,i}. \tag{5}$$

In order to approximate the $\phi_x(x_i)$ with high order of accuracy in smooth region and the essentially non-oscillatory property near discontinuities, the nonlinear weights $\omega_s(s=0,1,2)$ are computed as

$$\omega_s=\frac{\alpha_s}{\alpha_0+\alpha_1+\alpha_2},\quad \alpha_s=\frac{d_s}{(\epsilon+\beta_s)^2},\quad s=0,1,2, \tag{6}$$

where the small positive parameter ϵ is introduced to prevent the denominator becoming zero, the linear weights $d_s(s=0,1,2)$ are given as $d_0=\frac{1}{10}$, $d_1=\frac{6}{10}$, and $d_2=\frac{3}{10}$, and the smoothness indicators $\beta_s(s=0,1,2)$ are used to measure the smoothness of ϕ_x on the sth substencil and designed as follow [7,8]:

$$\begin{cases}\beta_0=\dfrac{1}{4}(\phi_{i-3}-5\phi_{i-2}+7\phi_{i-1}-3\phi_i)^2+\dfrac{13}{12}(\phi_{i-3}-3\phi_{i-2}+3\phi_{i-1}-\phi_i)^2,\\ \beta_1=\dfrac{1}{4}(\phi_{i-2}-\phi_{i-1}-\phi_i+\phi_{i+1})^2+\dfrac{13}{12}(\phi_{i-2}-3\phi_{i-1}+3\phi_i-\phi_{i+1})^2,\\ \beta_2=\dfrac{1}{4}(3\phi_{i-1}-7\phi_i+5\phi_{i+1}-\phi_{i+2})^2+\dfrac{13}{12}(\phi_{i-1}-3\phi_i+3\phi_{i+1}-\phi_{i+2})^2.\end{cases} \tag{7}$$