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The effect of perturbed advection on a class of solutions of a non-linear reaction-diffusion equation

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ABSTRACT

In this work, the traveling wave solutions of a one-dimensional reaction-diffusion equation with advection are studied. The traveling wave solutions are obtained using the G'/G-expansion method. The shock thickness and spectral stability have been discussed for the obtained solution in the parameter interval. The essential spectra of the perturbed and linearized differential operator about the traveling antikink and kink solutions at the equilibrium states are obtained. The point spectrum is calculated using Evans function with Lie midpoint method and Magnus method. It is shown that, for a symmetric potential well, the traveling kink and antikink solutions which connect the stable equilibrium states of the system are stable. It is observed that the perturbation on the advection exhibits contrasting effect on the solution properties (shock thickness and the eigenvalue) of kink and antikink solutions. Variation of the reaction coefficient leads to instability of the solutions, unlike the diffusion coefficient which enhances the stability. On the other hand, the variation of reaction and diffusion coefficients show the monotonic effect on the shock thickness of the traveling kink and antikink solutions. This study is expected to be useful in analyzing the slow or fast invasion and stability of the population movement in different steady states.

1. Introduction

Mathematical models of a number of physical, chemical, and biological systems are described by the non-linear reaction-diffusion advection equations (NLRDAE). The general form of NLRDAE in one-dimensional spatial variable is

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial}{\partial x}\left(D_0 \frac{\partial u(x,t)}{\partial x}\right) + \delta_0 \frac{\partial u(x,t)}{\partial x} + \gamma_0 f(u(x,t)) \quad -\infty < x < \infty, \quad 0 \le t < \infty \tag{1}$$

where $D_0 = D_0(c_0, x, t, u, u_x)$ contributes to diffusion, $f(u)$ plays the role of reaction in the system and c_0, δ_0 and γ_0 are constants. The NLRDAE can be used to describe the dynamics of the ecological models [1,2], genetic models [3], gas dynamics, population distribution [4,5] etc. The chemotactic and population pressure models with bistable reaction kinetics are considered for the wave speed analysis through perturbation and Melnikov function [1,6–10]. The evolution of bacteria of the type Paenibacillus dendritiformis, modeled as NLRDAE, exhibits sharp traveling wave patterns [11].

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In the past decades, a variety of methods have been devised to construct traveling wave solutions for the nonlinear partial differential equations (NPDE). The following are a few from among those methods: Bäcklund transform [12], truncated Painlevé expansion [13,14], inverse scattering transform method [12,13], first integral method [15,16], Exp-function method [17], modified simple equation method [18], Weierstrass elliptic function method [19,20], extended simplest equation method [21], method based on Bäcklund transform and the properties of ordinary differential equation [22], Q-function method [23] and G'/G-expansion method [24–26]. Various types of wave solutions are characterized as follows [27,28]. If $u(\xi)$ (where $\xi = x - ct$, c is the wave speed) is a traveling wave solution of a NPDE, then $u(\xi + \kappa)$, $\kappa \in \mathbb{R}$ is also a traveling wave solution of the NPDE. If $c = 0$, then the solution is called the stationary wave solution. A solution $u(\xi)$ is called periodic traveling wave, if $u(\xi + l) = u(\xi)$ for all ξ, for some $l > 0$. If $\lim_{\xi\to\pm\infty} u(\xi) = u^{\pm}$, a wave solution $u(\xi)$ with $u^+ \neq u^-$ $(u^+ = u^-)$ is known as kink or antikink (pulse) solution.

A natural approach to the study of stability of a wave solution is to linearize the NPDE about the wave solution. The given solution is said to be spectrally stable if and only if $\Re(\lambda_k) \leq 0$ for all k. The solution is asymptotically stable if and only if $\Re(\lambda_k) < 0$ for all k. In an unbounded domain, the spectrum of the operator (L) consists of both discrete (point spectra) eigenvalues (including zero, due to translation invariance), and continuous spectrum (essential spectra). Instability of traveling waves are classified as convective and absolute [29–31]. A local perturbation (wave packet) is convectively unstable if it grows in time but convected away (non-zero group velocity) from the point of generation. If, on the other hand, the group velocity goes to zero, an unstable perturbation permeates the medium completely. The system is then said to be absolutely unstable.

The direction of the traveling wave solution reveals expansion or a shrink in which population studies. The speed of the moving profile has been predicted through an analytical expression [6], may be helpful to experimental biologists. The generalized 1-soliton solution have been obtained using solitary wave anzatz for a class of a one-dimensional nonlinear reaction-diffusion linear advection equation with constant coefficients arising in chemical reaction engineering [32]. The G'/G-expansion method and a solitary wave anzatz have been used to obtain the solutions of the nonlinear partial differential equations with constant coefficients and no advection [33]. In a NLRDAE, perturbations can appear in the advection, diffusion, or reaction term [4,6,34]. It is important to know how perturbations may affect solution stability and characteristics. To study the effects of perturbation, we first obtain the traveling wave solution of a one-dimensional reaction-diffusion equation with perturbed advection term. Then, we study the solution properties in terms of their shock thickness and the spectral values. The shock thickness l_s of a kink solution is obtained as $l_s = \frac{u^+ - u^-}{u'(0)}$ [35]. The characteristics of the shock front are useful in many fields including computational modeling of brain injury (Blast-induced neurotrauma) [36]. It is observed that the effects of the different NLRDAE parameters (diffusion coefficient, advection perturbation parameter, coefficient of reaction term) on the shock thickness and stable eigenvalues are qualitatively different.

This article is organized as follows. The basics of the G'/G-expansion method which is used to obtain the analytical wave solutions and the procedure to obtain spectral values in unbounded domain are explained in Section 2. In Section 3, traveling wave solutions are derived for the one-dimensional reaction-diffusion equation with first order perturbed advection term. Then, the spectral stability analysis through Evans function with the Lie midpoint and the Magnus methods for the traveling wave solutions is performed. In Section 4, the effect of various parameters on the shock thickness and eigenvalue loci is studied. The paper is concluded with Section 5.

2. Procedure for analytical traveling wave solutions and spectral stability analysis

2.1. G'/G-*expansion method*

The fundamental idea of the G'/G-expansion method is the construction of the exact wave solutions of a nonlinear PDE using the solution $G(\xi)$ of a linear second order ordinary differential equation (ODE). Suppose a nonlinear partial differential equation is given by

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots) = 0, \tag{2}$$

where $u = u(x, t)$ is the field variable, depend on the independent variables x and t.

1. Taking x and t into a single variable $\xi = x - ct$, and writing $u(x, t) = u(\xi)$ permits us reduce (2) to an ODE as
$$P(u, -cu', u', c^2u'', -cu'', u'', \ldots) = 0, \tag{3}$$
2. Suppose that the solution of ODE (3) can be expressed by a finite series in terms of $(\frac{G'(\xi)}{G(\xi)})$ as follows:
$$u(\xi) = \sum_{i=0}^{m} a_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i, \tag{4}$$
where $G(\xi)$ satisfies the second order linear ODE
$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{5}$$
and $a_m \neq 0, a_{m-1}, \ldots, a_0, \lambda$ and μ are constants and $\lambda, \mu \in \mathbb{R}$. The value of m in the series (4) can be obtained from the homogeneous balance between the highest order derivatives and nonlinear terms in (3).