Applied Mathematics and Computation 290 (2016) 46–55

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

A unified model for solving a system of nonlinear equations

Qifang Su*

School of Mathematics and Information Engineering, Taizhou University, Linhai, Zhejiang 317000, PR China

ARTICLE INFO

MSC:
65H10

Keywords:
System of nonlinear equations
Newton's method
Order of convergence
Efficiency

ABSTRACT

In this paper, we suggest and analyze a unified model of two-step iterative method for solving a system of nonlinear equations. We prove that the unified model achieves fourth order convergence. Several numerical examples are given to illustrate the efficiency and the performance of the new iterative methods. From the comparison with some known methods it is observed that the present methods are of practical utility.

1. Introduction

Consider the system of nonlinear equations

$$\begin{cases} f_1(x_1, x_2, \dots, x_n) = 0, \\ f_2(x_1, x_2, \dots, x_n) = 0, \\ \vdots \\ f_n(x_1, x_2, \dots, x_n) = 0, \end{cases} \tag{1}$$

where each function f_i maps a vector $\mathbf{x} = (x_1, x_2, \dots, x_n)^T$ of the n-dimensional space R^n into the real line R. The system (1) can be written in the form

$$F(\mathbf{x}) = 0, \tag{2}$$

where $F(\mathbf{x}) = (f_1(\mathbf{x}), f_2(\mathbf{x}), \dots, f_n(\mathbf{x}))^T$.

Systems of nonlinear equations are of immense importance for applications in many sciences and engineering [1–3]. For a given nonlinear function F: $D \subset R^n \to R^n$, to find a vector $\mathbf{r} = (r_1, r_2, \dots, r_n)^T$ such that $F(\mathbf{r}) = 0$. It is well known, one of the basic procedures for solving the system of nonlinear equations $F(\mathbf{x}) = 0$, is the quadratically convergent Newton's method [1]:

$$\mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - F'(\mathbf{x}^{(k)})^{-1} F(\mathbf{x}^{(k)}), \tag{3}$$

where $F'(\mathbf{x})^{-1}$ is the inverse of the first Fréchet derivative of the function $F(\mathbf{x})$.

Recently, several iterative methods have been developed to solve the system of nonlinear equations $F(\mathbf{x}) = 0$, by using essentially Taylors polynomial [2,8], decomposition [6,9–11], homotopy perturbation method [12], quadrature formulas [13–20] and other techniques [21–25]. For example, Montazeri et al. considered a three-step structure using a Jarratt-type method as the predictor, while the corrector step would be designed as if no new Fréchet derivative has been used [7]. Noor

* Tel.: +86 57685137058.
E-mail address: suqf_tzc@163.com

http://dx.doi.org/10.1016/j.amc.2016.05.047

and Waseem suggested and analyzed two new two-step iterative methods *NR*1 and *NR*2 by using quadrature formulas and proved that these new methods have cubic convergence [5].

$$NR1: \begin{cases} \mathbf{y_k} = \mathbf{x_k} - \mathbf{F}'(\mathbf{x_k})^{-1}\mathbf{F}(\mathbf{x_k}), \\ \mathbf{x}_{k+1} = \mathbf{x}_k - 4\left[F'(\mathbf{x}_k) + 3F'\left(\frac{\mathbf{x}_k+2\mathbf{y}_k}{3}\right)\right]^{-1} F(\mathbf{x}_k). \end{cases} \tag{4}$$

$$NR2: \begin{cases} \mathbf{y_k} = \mathbf{x_k} - \mathbf{F}'(\mathbf{x_k})^{-1}\mathbf{F}(\mathbf{x_k}), \\ \mathbf{x}_{k+1} = \mathbf{x}_k - 4\left[3F'\left(\frac{2\mathbf{x}_k+\mathbf{y}_k}{3}\right) + F'(\mathbf{y}_k)\right]^{-1} F(\mathbf{x}_k). \end{cases} \tag{5}$$

Motivated and inspired by the on-going activities, we suggest a unified model of two-step iterative method for solving a system of nonlinear equations and analyze its convergence.

The paper is organized as follows. In Section 2, the new type method is developed and analyzed. In Section 3, a discussion on the efficiency index of the contribution with comparison to the other iterative methods is given. In Section 4, some numerical examples are given to illustrate the efficiency of the new methods. And finally, some conclusions are drawn in Section 5.

2. Development of the methods

From the schemes (4) and (5) and the motivation of improving the convergence order without second or higher derivatives, we consider the unified model as follows

$$\begin{cases} \mathbf{y^{(k)}} = \mathbf{x^{(k)}} - \mathbf{F}'(\mathbf{x^{(k)}})^{-1}\mathbf{F}(\mathbf{x^{(k)}}), \\ \mathbf{x}^{(k+1)} = \mathbf{x}^{(k)} - \left[\sum_{i=1}^{s} c_i F'(\alpha_i \mathbf{x}^{(k)} + \beta_i \mathbf{y}^{(k)})\right]^{-1} \left[\sum_{j=1}^{t} d_j F(\gamma_j \mathbf{x}^{(k)} + \varphi_j \mathbf{y}^{(k)})\right], \end{cases} \tag{6}$$

where $\alpha_i + \beta_i = 1,\ \gamma_j + \varphi_j = 1$ and $\sum_{i=1}^{s} c_i \neq 0$.

In order to explore the convergence properties of the new method (6), we first recall the following result of Taylor's expression on vector functions.

Lemma 1 [4]. *Let* $\mathbf{F}: D \subseteq R^n \to R^n$ *be p-times Fréchet differentiable in a convex set* $D \subseteq R^n$*, then for any* $\mathbf{x}, \mathbf{h} \in R^n$*, the following expression holds:*

$$F(\mathbf{x}+\mathbf{h}) = F(\mathbf{x}) + F'(\mathbf{x})\mathbf{h} + \frac{1}{2!}F''(\mathbf{x})\mathbf{h}^2 + \frac{1}{3!}F'''(\mathbf{x})\mathbf{h}^3 + \cdots + \frac{1}{(p-1)!}F^{(p-1)}(\mathbf{x})\mathbf{h}^{p-1} + \mathbf{R}_p, \tag{7}$$

where

$$\|\mathbf{R}_p\| \le \frac{1}{p!} \sup_{0\le t\le 1} \|F^{(p)}(\mathbf{x}+t\mathbf{h})\| \|\mathbf{h}\|^p \text{ and } \mathbf{h}^p = (\mathbf{h}, \mathbf{h}, \overset{p}{\ldots}, \mathbf{h}).$$

Theorem 1. *Let* $\mathbf{F}: D \subseteq R^n \to R^n$ *be sufficiently Fréchet differentiable at each point of an open convex neighborhood D containing the simple root* $\mathbf{r}$ *of* $F(\mathbf{x}) = 0$*. Then the sequence* $\{\mathbf{x}^{(k)}\}$ *obtained by using the iterative expression of method (6) converges to* $\mathbf{r}$ *with convergence order four under some suitable parameters.*

Proof. Let $\mathbf{e}^{(k)} = \mathbf{x}^{(k)} - \mathbf{r}$. From the second step in (6), we have

$$\left[\sum_{i=1}^{s} c_i F'(\alpha_i \mathbf{x}^{(k)} + \beta_i \mathbf{y}^{(k)})\right]\mathbf{e}^{(k+1)} = \left[\sum_{i=1}^{s} c_i F'(\alpha_i \mathbf{x}^{(k)} + \beta_i \mathbf{y}^{(k)})\right]\mathbf{e}^{(k)} - \left[\sum_{j=1}^{t} d_j F(\gamma_j \mathbf{x}^{(k)} + \varphi_j \mathbf{y}^{(k)})\right]. \tag{8}$$

The Taylor's expansion for $F(\mathbf{x})$ about $\mathbf{x}^{(k)}$ is

$$\begin{aligned} F(\mathbf{x}) &= F(\mathbf{x}^{(k)}) + F'(\mathbf{x}^{(k)})(\mathbf{x}-\mathbf{x}^{(k)}) + \frac{1}{2!}F''(\mathbf{x}^{(k)})(\mathbf{x}-\mathbf{x}^{(k)})^2 \\ &\quad + \frac{1}{3!}F'''(\mathbf{x}^{(k)})(\mathbf{x}-\mathbf{x}^{(k)})^3 + \frac{1}{4!}F^{(4)}(\mathbf{x}^{(k)})(\mathbf{x}-\mathbf{x}^{(k)})^4 + O(\|\mathbf{x}-\mathbf{x}^{(k)}\|^5). \end{aligned} \tag{9}$$

By using $F(\mathbf{r}) = 0$, we obtain

$$F(\mathbf{x}^{(k)}) = F'(\mathbf{x}^{(k)})\mathbf{e}^{(k)} - \frac{1}{2!}F''(\mathbf{x}^{(k)})(\mathbf{e}^{(k)})^2 + \frac{1}{3!}F'''(\mathbf{x}^{(k)})(\mathbf{e}^{(k)})^3 - \frac{1}{4!}F^{(4)}(\mathbf{x}^{(k)})(\mathbf{e}^{(k)})^4 + O(\|\mathbf{e}^{(k)}\|^5), \tag{10}$$

and