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On symmetries and conservation laws of a Gardner equation involving arbitrary functions

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ABSTRACT

In this work we study a generalized variable-coefficient Gardner equation from the point of view of Lie symmetries in partial differential equations. We find conservation laws by using the multipliers method of Anco and Bluman which does not require the use of a variational principle. We also construct conservation laws by using Ibragimov theorem which is based on the concept of adjoint equation for nonlinear differential equations.

1. Introduction

Nonlinear equations with variable coefficients have become increasingly important in recent years because these describe many nonlinear phenomena more realistically than equations with constant coefficients. The Gardner equation, for instance, is used in different areas of physics, such as fluid dynamics, plasma physics, quantum field theory, and it also describes a variety of wave phenomena in plasma and solid state.

In this paper, we consider the variable-coefficient Gardner equation with nonlinear terms given by

$$u_t + A(t)uu_x + C(t)u^2u_x + B(t)u_{xxx} + Q(t)u = 0, \tag{1}$$

where $A(t) \neq 0$, $B(t) \neq 0$, $C(t) \neq 0$ and $Q(t)$ are arbitrary smooth functions of t.

In [10], for $A(t) = 1$ and $C(t) = 0$, the optimal system of one-dimensional subalgebras was obtained. In [11], some conservation laws for Eq. (1) were constructed for some special forms of the functions $B(t)$ and $Q(t)$. Lie symmetries of Eq. (1) when $Q(t) = 0$, were derived in [15]. The classification of Lie symmetries obtained in [15] was enhanced in [19] by using the general extended equivalence group. In [23], adding to Eq. (1) the term $E(t)u_x$, where $E(t)$ is an arbitrary smooth function of t, the authors found new exact non-traveling solutions, which include soliton solutions, combined soliton solutions, triangular periodic solutions, Jacobi elliptic function solutions and combined Jacobi elliptic function solutions of Eq. (1). Soliton solutions of Eq. (1) were obtained in [20] transforming the equation to an homogeneous equation when $Q(t) = 0$ and a forcing term $R(t)$ has been added. Finally, in [9], exact solutions were obtained by using the general mapping deformation method adding a new term $E(t)u_x$ and a forcing term $R(t)$ to Eq. (1), where $E(t)$ and $R(t)$ are arbitrary smooth functions of t.

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Lie symmetries, in general, symmetry groups, have several applications in the context of nonlinear differential equations. It is noteworthy that they are used to obtain exact solutions and conservation laws of partial differential equations [5–7,16,22].

In [3] Anco and Bluman gave a general treatment of a direct conservation law method for partial differential equations expressed in a standard Cauchy–Kovaleskaya form

$$u_t = G(x, u, u_x, u_{xx}, \dots, u_{nx}).$$

Nontrivial conservation laws are characterized by a multiplier λ, which has no dependence on u_t and all derivatives of u_t, satisfying

$$\hat{E}[u](\lambda u_t - \lambda G(x, u, u_x, u_{xx}, \dots, u_{nx})) = 0.$$

Here

$$\hat{E}[u] := \frac{\partial}{\partial u} - D_t \frac{\partial}{\partial u_t} - D_x \frac{\partial}{\partial u_x} + D_x^2 \frac{\partial}{\partial u_{xx}} + \cdots .$$

The conserved density T^t must satisfy

$$\lambda = \hat{E}[u]T^t,$$

and the flux T^x is given by

$$T^x = -D_x^{-1}(\lambda G) - \frac{\partial T^t}{\partial u_x} G + G D_x\left(\frac{\partial T^t}{\partial u_{xx}}\right) + \cdots .$$

In [13], Ibragimov introduced a general theorem on conservation laws which does not require the existence of a classical Lagrangian and it is used based on the concept of an adjoint equation for nonlinear equations. In [14], Ibragimov generalized the concept of linear self-adjointness by introducing the concept of nonlinear self-adjointness of differential equations. This concept has been recently used for constructing conservation laws [7,18].

The aim of this work is to obtain Lie symmetries of Eq. (1) and construct conservation laws by using both methods, the direct method proposed by Anco and Bluman [2,3] and Ibragimov theorem [13]. We have studied Lie symmetries of equation (1) for cases $Q \neq 0$ and $Q = 0$. In order to obtain conservation laws using Ibragimov theory we have determined the subclasses of Eq. (1) which are nonlinearly self-adjoint.

2. Classical symmetries

To apply the Lie classical method to Eq. (1) we consider the one-parameter Lie group of infinitesimal transformations in (x, t, u) given by

$$\begin{aligned} x^* &= x + \epsilon \xi(x, t, u) + O(\epsilon^2), \\ t^* &= t + \epsilon \tau(x, t, u) + O(\epsilon^2), \\ u^* &= u + \epsilon \eta(x, t, u) + O(\epsilon^2), \end{aligned}$$

where ϵ is the group parameter. We require that this transformation leaves invariant the set of solutions of Eq. (1). This yields an overdetermined, linear system of differential equations for the infinitesimals $\xi(x, t, u)$, $\tau(x, t, u)$ and $\eta(x, t, u)$. The associated Lie algebra of infinitesimal symmetries is formed by the set of vector fields of the form

$$\mathbf{v} = \xi(x, t, u)\partial_x + \tau(x, t, u)\partial_t + \eta(x, t, u)\partial_u. \tag{2}$$

Invariance of Eq. (1) under a Lie group of point transformations with infinitesimal generator (2) leads to a set of 18 determining equations. By simplifying this system we obtain that $\xi = \xi(x, t)$, $\tau = \tau(t)$, and $\eta = \eta(x, t, u)$ are related by the following conditions:

$$\begin{aligned} &\eta_{uu} = 0, \quad \eta_{ux} - \xi_{xx} = 0, \quad \eta_{uuu} = 0, \quad \eta_{uux} = 0, \quad -\tau B_t - \tau_t B + 3\xi_x B = 0, \quad \tau u B Q_t - \tau u B_t Q - \eta_u u B Q + 3\xi_x u B Q \\ &\quad + \eta B Q + \eta_x u^2 B C + \eta_{xxx} B^2 + \eta_x u A B + \eta_t B = 0, \quad \tau u^3 B C_t - \tau u^3 B_t C + \xi_x u^3 B C + 2\eta u^2 B C - \tau u^2 A B_t + 3\eta_{uxx} u B^2 \\ &\quad - \xi_{xxx} u B^2 + \tau u^2 A_t B + 2\xi_x u^2 A B + \eta u A B - \xi_t u B = 0. \end{aligned} \tag{3}$$

In order to find Lie symmetries of the equation, we distinguish two cases: $Q \neq 0$ and $Q = 0$.

Case 1. $Q \neq 0$.

For the sake of simplicity, in Case 1 we shall consider $C(t) = 1$, obtaining the following symmetries

$$\xi = k_1 x + \beta, \quad \tau = \tau(t), \quad \eta = \frac{\beta_t}{A} + (k_1 + \alpha)u$$

where $A = A(t)$, $B = B(t)$, $Q = Q(t)$, $\alpha = \alpha(t)$, $\beta = \beta(t)$ and $\tau = \tau(t)$ must satisfy the following conditions:

$$(3k_1 - \tau_t)B - \tau B_t = 0, \tag{4}$$