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Finite volume element method and its stability analysis for analyzing the behavior of sub-diffusion problems

K. Sayevand*, F. Arjang

Faculty of Mathematical Sciences, Malayer University, P.O. Box 16846-13114, Malayer, Iran

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ABSTRACT

In this paper, we analyze the spatially semi-discrete piecewise linear finite volume element method for the time fractional sub-diffusion problem in two dimensions, and give an approximate solution of this problem. At first, we introduce bilinear finite volume element method with interpolated coefficients and derive some error estimates between exact solution and numerical solution in both finite element and finite volume element methods. Furthermore, we use the standard finite element Ritz projection and also the elliptic projection defined by the bilinear form associated with the variational formulation of the finite volume element method. Finally, some numerical examples are included to illustrate the effectiveness of the new technique.

1. Introduction

Many of phenomena can be modeled mathematically as fractional ordinary and partial differential equations [1–7], and these fractional models may be used more accurately than the classical integer order differential equations. In recent decades, many researchers have considered the fractional diffusion models to describe some natural phenomena such as diffusion processes and chaotic dynamics [8,9] such as magnetic plasma [10], dissipation [11], turbulent flow [12], electron transportation [13], etc.

Fractional diffusion equation as a class of time fractional partial differential equations, received considerable attention by many researchers from both theoretical and numerical points of view. The analytical solutions of these fractional models have obtained using Laplace transforms [14,15], Green's functions [16] or Fourier-Laplace transforms [17]. Since analytical solutions are thoroughly not accessible, except when initial and boundary conditions are simple, thus the numerical simulation has an important role in analysis of fractional diffusion equations.

Numerical solution of fractional diffusion equations were obtained by some numerical schemes such as finite difference methods, spectral methods and finite element methods. Lin and Xu [18] applied the finite difference method for the time fractional diffusion equation and investigated the stability condition. Cui [19] considered a compact finite difference method for fractional diffusion equation. Chen et al. [20] used an implicit finite difference method for fractional percolation equation. Zhang et al. [21] employed the finite difference method to solve the fractional diffusion equation with nonuniform mesh. Li and Xu [22] propounded a time-space spectral method with some error analysis, and a least squares spectral method was utilized by Carella and Dorago [23] for a fractional advestion–dispersion equation.

* Corresponding author. Tel.: +98813339881.
E-mail addresses: ksayehvand@malayeru.ac.ir, ksayehvand@iust.ac.ir (K. Sayevand), arjang21481@gmail.com (F. Arjang).

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Firstly, Ervin and Roop [24,25] applied finite element methods to solve the fractional advection–dispersion equation. A finite element method was employed by Ford et al. [26] for time fractional partial differential equations with stability analysis. Zhang et al. [27] considered a finite element method for one-dimensional time fractional Tricomi-type equations. Mustapha and McLean [28] proposed a finite element method for a fractional diffusion equation and uniform convergence was carried out. Although, various finite difference methods and finite element methods have been constructed to treat such problems, and global error estimates were also derived for one, two and three dimensional problems, but as far as we know, there has been very little work on the finite volume element discretization. Finally, for more details about some efficient numerical approaches for differential equations of fractional order and their applications the reader is advised to consult the results of research works presented in [29–35].

This study applies the finite volume element method to solve the fractional sub-diffusion equation with initial and boundary conditions as

$$\begin{cases} \dfrac{\partial^\alpha u}{\partial t^\alpha}(\mathbf{x},t) = \beta\, \Delta u(\mathbf{x},t) + f(\mathbf{x},t), & \text{in } \Omega \times [0,T], \\ u(\mathbf{x},t) = 0, & \text{on } \partial\Omega \times [0,T], \\ u(\mathbf{x},0) = v(\mathbf{x}), & \text{in } \Omega, \end{cases} \quad (1)$$

where the data of the problem, v and f are sufficiently smooth functions. Also, v is the initial value of u, f is the source term and $\beta > 0$ is the fractional diffusion coefficient. For $\mathbf{x} = (x_1, \dots, x_d)$, $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$ is the Laplacian operator and Ω is a bounded domain in space R^d with boundary $\partial\Omega$. Also, the fractional derivative $\frac{\partial^\alpha}{\partial t^\alpha}$ is in the Caputo sense defined by

$$\frac{\partial^\alpha u(x,y,t)}{\partial t^\alpha} = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{u_\tau(x,y,\tau)}{(t-\tau)^\alpha} d\tau, \quad 0 < \alpha < 1, \quad (2)$$

where $\Gamma(\alpha)$ is the well-known Gamma function.

The main purpose of this study is design and analyze a method using finite volume element method in space discretization for the two dimensional fractional model of (1), i.e., $\mathbf{x} = (x_1, x_2) \in \Omega \subset R^2$. Then, to define a fully discrete scheme, we combine the finite volume element method in space with a backward finite difference method in time discretization.

The finite volume element method is a generalization of the finite difference and finite element methods, in fact it lies between these two methods. It also named as box method or spectral volume method [36–39]. As we know, the finite volume element method uses a volume integral formulation of the differential equations, i.e., in addition to the primary partition (finite element partition) on the domain that approximates the exact solution, we introduce a dual partition as control volumes to discretize the equation. These control volumes have flexibility to handle the complicated domains and keep the conservation law of energy and mass; thus they have intensively considered to solve the partial differential equations. Another reason of choice the finite volume element method is that error analysis in this approach is similar in sense to tools developed for the error estimates of finite element method. Therefore, we use this framework to solve the problem (1). For some earlier results in using finite volume element method, one can refer to Cai [40] and Suli [41]. Then, Chou and Li [42] considered the desired method for elliptic and parabolic problems and obtained some error analysis. Chatzipantelidi et al. [43] applied this method for parabolic equations in convex polygonal domains and investigated its error estimates. Yang et al. [44] proposed this method to solve the time-space fractional diffusion equation in two dimensions, and Hejazi et al. [45] investigated error analysis of a finite volume method for the space fractional advection–dispersion equation, just to mention a few.

The assortment of the rest of this paper is as follows: in the next section, we introduce the finite volume element method and in Section 3, we will obtain a numerical solution for problem (1) helping by control volumes and finite element method. In the following section, Section 4, we study both finite element and finite volume methods, and give some error estimates corresponding to the initial data of the problem. Some numerical examples are given to illustrate the applicability of the new method in Section 5 and finally, Section 6 is devoted to some conclusions.

2. Finite volume element method

This section has been allocated to some notations in subject Sobolev spaces, primary and dual partitions on the considered domain Ω, the finite dimensional spaces and finite volume element method with some primary results.

Let $\mu = (\mu_1, \mu_2)$, where the μ_i are non-negative integers. For a given function v: $\mathbf{R}^2 \to \mathbf{R}$, its partial derivatives of order $|\mu| = \mu_1 + \mu_2$ can be written as

$$D^\mu v = \frac{\partial^{|\mu|} v}{\partial x_1^{\mu_1} \partial x_2^{\mu_2}}, \quad (x_1, x_2) \in \mathbf{R}^2. \quad (3)$$

For $\Omega \subset R^2$ a domain with smooth boundary $\partial\Omega$ and $\mathbf{x} \in \Omega$, we denote the space $L_2(\Omega)$ of functions that are square integrable with scalar product and norm

$$(u, v) = (u, v)_{L_2(\Omega)} = \int_\Omega u\, v\, d\mathbf{x}, \quad \|v\|_0 = \|v\|_{L_2(\Omega)} = \left(\int_\Omega v^2 d\mathbf{x}\right)^{\frac{1}{2}}. \quad (4)$$