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A modified regularization method for a Cauchy problem for heat equation on a two-layer sphere domain

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ABSTRACT

In this paper, we study a non-characteristic Cauchy problem for a radially symmetric inverse heat conduction equation in a two-layer domain. This is a severely ill-posed problem in the sense that the solution (if it exists) does not depend continuously on the data. It is well-known that the classical Tikhonov regularization solutions are too smooth and the approximate solutions may lack details that might be contained in the exact solutions. Combining Fourier transform technique with a modified version of the classical Tikhonov regularization, we obtain a regularized solution which is stably convergent to the exact solution with a sharp error estimate.

1. Introduction

In many real physical applications, it is necessary to determine the temperature or heat flux on the surface of a body from measured temperature at some fixed locations inside the body [1]. This so-called inverse heat conduction problem (IHCP) has become an interesting subject recently [2–5]. In the IHCP, it is required to recover the surface temperature and heat flux on an inaccessible boundary from the measurements on an accessible boundary of the body. This will lead to a non-characteristic Cauchy problem of heat equation, which is a well-known ill-posed problem [6,7]. Theoretical justifications and computational methods related to the Cauchy problem of radially symmetric heat equation have been discussed by many authors [8,9]. Murio has presented the estimation of the boundary temperature on a sphere from measurements at its center [10]. The problem of determining transient temperature distribution in a composite medium consisting of several layers in contact arise in numerous applications from engineering, e.g., for the direct problems, let's refer to [11]; for the inverse problems, let's refer to [12–18]. To the knowledge of the authors, however, there are still very few works on the regularization error analysis for the Cauchy problem of heat equation in multi-layer domain. In order to solve the IHCP in multi-layer bodies, it is natural to solve the IHCP separately in each layer.

In this paper, we are interested in establishing an analytical solution method for the solution of the Cauchy problem in multi-layer domain. By using the variable substitution and Fourier transform techniques, we first obtain an analytical solution in the frequency domain. To tackle the ill-posedness of the problem, we then employ a modified Tikhonov regularization technique based on the analytical solution to stably reconstruct the solution to the Cauchy problem. For illustration, we presents the regularization error analysis based on the analytical solution for the IHCP in a two-layer domain. Furthermore, we derive the modified version of classical Tikhonov regularization method with proofs on the error estimates.

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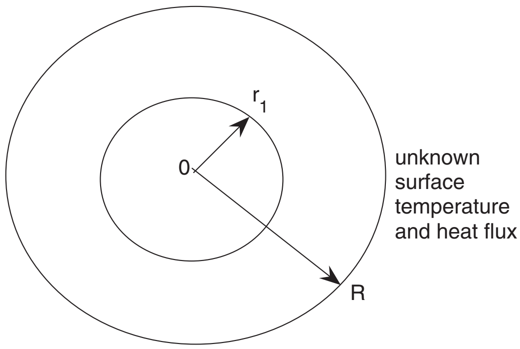

Fig. 1. The mathematical model in a two-layer sphere domain.

The paper is organized as follows. In Section 2, we first give the formulation of the Cauchy problem of radially symmetric heat equation in multi-layer domain. The modified Tikhonov regularization solution is then obtained with error estimates proven in Section 3.

2. Description of the problem

Consider a two-layer sphere that consists of the first layer in $0 \le r \le r_1$ and the second layer in $r_1 \le r \le R$. The two layers are in perfect thermal contact at $r = r_1$, as displayed in Fig. 1.

Let $k_1, k_2 > 0$ be the thermal conductivities and $\alpha_1, \alpha_2 > 0$ be the thermal diffusivities of the first and the second layer, respectively. The temperature distributions in the first and the second layers, denoted by $v_1(x,t)$ and $v_2(x,t)$, respectively, satisfy the following conditions in the two domains $D_1 := \{r|0 < r \le r_1\}$ and $D_2 := \{r|r_1 \le r \le R\}$:

$$\begin{cases} \dfrac{\partial v_1}{\partial t} = \alpha_1\left(\dfrac{2}{r}\dfrac{\partial v_1}{\partial r} + \dfrac{\partial^2 v_1}{\partial r^2}\right), & 0 < r < r_1, t > 0, \\ v_1(0,t) = F(t), & t > 0, \\ \dfrac{\partial v_1}{\partial r}(0,t) = 0, & t > 0. \end{cases} \tag{2.1}$$

and

$$\begin{cases} \dfrac{\partial v_2}{\partial t} = \alpha_2\left(\dfrac{2}{r}\dfrac{\partial v_2}{\partial r} + \dfrac{\partial^2 v_2}{\partial r^2}\right), & r_1 < r < R, t > 0, \\ v_2(r_1,t) = v_1(r_1,t), & t > 0, \\ k_2\dfrac{\partial v_2}{\partial r}(r_1,t) = k_1\dfrac{\partial v_1}{\partial r}(r_1,t), & t > 0, \end{cases} \tag{2.2}$$

subject to homogeneous the initial conditions

$$v_1(r,0) = v_2(r,0) = 0, \qquad 0 < r < R. \tag{2.3}$$

Throughout this paper, we suppose that $k_1/k_2 = c$ is a constant and for simplicity $c = 1$, the exact data $F(t) \in L^2(0,\infty)$ and thus it is natural to assume that, for any fixed $r \in [0,R]$, the solution $v_1(r,\cdot), v_2(r,\cdot)$ and their derivatives $\frac{\partial v_1(r,\cdot)}{\partial r}, \frac{\partial v_2(r,\cdot)}{\partial r}$ belong to $L^2(0,\infty)$.

The inverse Cauchy problem is then to determine the solutions $v_2(r,t)$ and $\frac{\partial v_2(r,t)}{\partial r}$ for $r_1 \le r \le R$ in the space $L^2(0,\infty)$ from the measured data $F(t) \in L^2(0,\infty)$ and the insulated condition at the accessible boundary $r = 0$. Usually the measured data $F(t)$ contains error that gives $F^\delta(\cdot) \in L^2(0,\infty)$ satisfying

$$\| F^\delta(\cdot) - F(\cdot) \| \le \delta, \tag{2.4}$$

where the constant $\delta \ge 0$ represents a bound on the measurement error and $\|\cdot\|$ denotes the L^2-norm. Assume that there exists a constant $E \ge 0$ so that the following a-priori bound exists for the solution $v_2(R,t)$ of the problem (2.2):

$$\| v_2(R,\cdot) \| \le E. \tag{2.5}$$

Under the variable transformations

$$u(r,t) = rv_1(r,t), \tag{2.6}$$

$$\frac{\partial u}{\partial r}(r,t) = r\frac{\partial v_1}{\partial r}(r,t) + v_1(r,t), \tag{2.7}$$