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Solving some generalized Vandermonde systems and inverse of their associate matrices via new approaches for the Binet formula

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ABSTRACT

The aim of this paper is to exhibit new expressions for the Binet formula, without solving any type of Vandermonde systems. This allows us to establish a tractable formula for generalized Fibonacci sequences. As a consequence, new solutions of the Vandermonde systems and some special generalized Vandermonde systems are provided, moreover the inverses of their associated matrices are explicated. Some new results are formulated, also others are recovered and generalized. Finally, comparisons with other methods are broached and some numerical examples are supplied.

1. Introduction

In this paper we investigate some new expressions of the Binet formula for generalized Fibonacci sequences, obtained from their combinatorial expression. Notably, we show how to apply the results on the Binet formula for purpose to get solutions of the usual Vandermonde and some special generalized Vandermonde systems. One of the beauty of our study is to derive in a simple way, without alluding to the current approaches used in the literature, the compact representations of the inverses of the associated Vandermonde and special generalized Vandermonde matrices.

For reasons of conciseness, let first recall what follows. For a given $a_0, a_1, \dots, a_{r-1}$ with $a_{r-1} \neq 0$, and $v_0, v_1, \dots, v_{r-1}$ in $\mathbb{R}$ or $\mathbb{C}$, the sequence $\{V_n\}_{n \geq 0}$ defined by $V_n = v_n$ for $0 \leq n \leq r-1$ and the linear recurrence relation of order r,

$$V_{n+1} = a_0 V_n + \cdots + a_{r-1} V_{n-r+1}, \quad \text{for } n \geq r-1, \tag{1}$$

is called *weighted r-generalized Fibonacci sequence* or simply *r-generalized Fibonacci sequence* (see [9,13] for example). In the sequel we shall refer to them as *sequence (1)*. It is worth noting that sequences (1) are largely studied in the literature, owing to their theoretical and applied aspect in various fields of mathematics and applied sciences. The polynomial $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-2} z - a_{r-1}$, called *the characteristic polynomial of sequences (1)*, plays an important role in the determination of the explicit expression of the general term V_n, by considering $\lambda_1, \lambda_2, \dots, \lambda_s$ the (characteristic) roots of $P(z)$, with multiplicities $m_1, m_2 \dots, m_s$ (respectively). Indeed, it is well known that the compact formula of V_n is given by the Binet

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formula,

$$V_n = \sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} \beta_{i,j} n^j \right) \lambda_i^n, \tag{2}$$

for $n \geq 0$, where the $\beta_{i,j}$ are determined uniquely by the initial conditions $\{v_j\}_{0 \leq j \leq r-1}$ (see [9,13,20], for example).[2] More precisely, we can determine the $\beta_{i,j}$ by solving the system of r linear equations

$$\sum_{i=1}^{s} \left(\sum_{j=0}^{m_i-1} \beta_{i,j} n^j \right) \lambda_i^n = v_n, \quad n = 0, 1, \ldots, r-1. \tag{3}$$

Binet formula (2) has been expressed under another form in [7], by introducing the notion of their factorial Binet formula. However, the problem to determine the Binet formula (or factorial Binet) is still related to the resolution of the linear system (3). If $m_i = 1$ for every i, the linear system (3) is nothing else but the usual Vandermonde system largely studied in the literature (see for example [8,11,18,23] and references therein). If there exists some i such that $m_i \geq 2$, the system (3) represents a special generalized Vandermonde system, some particular cases have been examined in [15].

The preceding discussion exhibits the closed connection between the Binet formula and the two kind of Vandermonde systems. Usually, for establishing the Binet formula in the literature, we have to solve the Vandermonde systems (3) or computing the inverses of the associated matrices, which is a tough task in general. Indeed, the topic of inversion of Vandermonde and confluent Vandermonde matrices were especially popular since 1970, and it was subject of many publications. Recently, this field has received a regain of interests because of its various applications, particularly in the interpolation problems. Therefore, various numerical and theoretical approaches have been developed to success such inversion (see for instance [8,10,11,18,22,23] and references therein).

The primordial idea that triggers the study of our paper is the fact that the general term V_n of (1), can be also expressed under a combinatorial form (see for instance [2,12,14,17,19]). Notably, the more general form of this combinatorial form has been given in [17], where V_n is expressed in terms of some quantities labeled $\rho(n, r)$ (see Expression (5), here after), that will play a central role in this investigation. Such $\rho(n, r)$ are also related to the multivariate Fibonacci polynomials $H_n^{(r)}(a_0, \cdots, a_{r-1})$ of order r of Philippou as follows $\rho(n,r) = H_{n-r+1}^{(r)}(a_0, \cdots, a_{r-1})$ (see [2]). Moreover, this combinatorial form of (1), has been applied in the study of the powers and exponential of matrices in [1,3,4], where the $\rho(n, r)$ was given explicitly in terms of the roots $\lambda_1, \lambda_2, \ldots, \lambda_s$, of the characteristic polynomial $P(z) = z^r - a_0 z^{r-1} - \cdots - a_{r-2} z - a_{r-1}$, taking into account their multiplicities $m_1, m_2 \ldots, m_s$ (respectively). It ensues that the Binet formula (2) of the quantities $\rho(n, r)$, can be derived without solving any Vandermonde or special generalized Vandermonde systems (3).

The material of this paper is organized as follows. The aim of Section 2 is to establish some new expressions of the Binet formula for sequences (1), without appealing the approach reposed on the resolution of the linear system (3). Our method is merely based on the combinatorial expression of sequences (1). Section 3 is devoted to the application of these new expressions of the Binet formula of sequences (1), for solving the usual linear Vandermonde systems of equations. Therefore, explicit formula for the entries of the inverse of its associated matrices are also obtained directly. Illustrative examples are given. Moreover, a sketch comparison with two current methods and some numerical aspect of our results are presented. In Section 4 we examine the general case, when some multiplicities $m_i \geq 2$ $(1 \leq i \leq s)$ of the roots $\lambda_1, \ldots, \lambda_s$ are ≥ 2. We provide some new results on the special generalized Vandermonde systems (3), and we paraphrase some others of [15], even better we generalize them. A brief discussion and an illustrative numerical example are provided. Finally, concluding remarks and perspectives are given in Section 5.

2. New approaches upon the Binet formula of sequences (1)

Let $\{V_n\}_{n \geq 0}$ $(r \geq 2)$ be a sequence (1) of coefficients $a_0, a_1 \ldots, a_{r-1}$ $(a_{r-1} \neq 0)$ and initial values $V_0, V_1, \ldots, V_{r-1}$. For reason of clarity we recall that the combinatorial form of the general term V_n given by (1) has been established by various methods [2,12,14,17,19]. One of the more general compact form of the combinatorial form of V_n, has been provided in [17], using the connection between sequences (1) and the Markov chains. That is, when the coefficients $a_0, \ldots, a_{r-1}$ are non-negative real numbers of sum 1, it was shown in [17] that sequences (1) can be written as a matrix form $X = PX$, where $X = {}^t(V_0, V_1, \ldots, V_{r-1}, V_r, \ldots, V_n, \ldots)$ and P is a stochastic matrix, under the condition $a_0 + a_1 + \cdots + a_{r-1} = 1$ $(a_{r-1} \neq 0)$. Properties of the associated Markov chains lead to get,

$$V_n = \rho(n,r) A_0 + \rho(n-1,r) A_1 + \cdots + \rho(n-r+1,r) A_{r-1} \tag{4}$$

for every $n \geq r$, with $A_m = a_{r-1} V_m + \cdots + a_m V_{r-1}$ and

$$\rho(n,r) = \sum_{k_0+2k_1+\cdots+rk_{r-1}=n-r} \frac{(k_0+\cdots+k_{r-1})!}{k_0! k_1! \cdots k_{r-1}!} a_0^{k_0} a_1^{k_1} \cdots a_{r-1}^{k_{r-1}} \tag{5}$$

[2] In this paper, we adopt the convention that $0^0 = 1$.