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On the solutions of a system of difference equations with maximum☆

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ABSTRACT

In this paper, we study the following max-type system of difference equations

$$\begin{cases} x_n = \max\left\{\dfrac{1}{x_{n-m}}, \min\left\{1, \dfrac{A}{y_{n-r}}\right\}\right\}, \\ y_n = \max\left\{\dfrac{1}{y_{n-m}}, \min\left\{1, \dfrac{B}{x_{n-t}}\right\}\right\}, \end{cases} \quad n \in \mathbb{N}_0$$

where $A, B \in (0, +\infty)$, $m, r, t \in \{1, 2, \dots\}$ with $r \neq m$ and $t \neq m$. We show that every solution of this system with the initial values $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \dots, x_{-1}, y_{-1} \in (0, +\infty)$ is eventually periodic with period $2m$, where $d = \max\{m, r, t\}$.

1. Introduction

Recently, there has been a great interest in studying difference equations and systems which do not stem from differential ones (see, e.g., [1–35]). A class of difference equations that has attracted recent attention is the class of, so called, max-type difference equations (see, e.g., [1,3–7,9–11,13–19,23,31–34]). On the other hand, some concrete classes of nonlinear systems of difference equations have also attracted some recent attention (see, e.g., [2,8,22,25,26,30]). Some of recent papers belong to the both areas (see, e.g., [12,20,21,24,27–29,35]). For example, Stević [20] obtained the general solution for the following max-type system of difference equations

$$\begin{cases} x_{n+1} = \max\left\{\dfrac{\alpha}{x_n}, \dfrac{y_n}{x_n}\right\}, \\ y_{n+1} = \max\left\{\dfrac{\alpha}{y_n}, \dfrac{x_n}{y_n}\right\}, \end{cases} \quad n \in \mathbb{N}_0 \tag{1.1}$$

where $\alpha \in \mathbb{R}_+ \equiv (0, +\infty)$ and the initial values $x_0, y_0 \in [\alpha, +\infty)$ and $y_0/x_0 \geq \max\{\alpha, 1/\alpha\}$.

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In 2012, Stević [21] studied the following max-type system of difference equations

$$\begin{cases} y_n^{(1)} = \max_{1\le i\le m_1}\left\{f_{1i}(y^{(1)}_{n-k^{(1)}_{i,1}}, y^{(2)}_{n-k^{(1)}_{i,2}}, \ldots, y^{(l)}_{n-k^{(1)}_{i,l}}, n), y^{(1)}_{n-s}\right\}, \\ y_n^{(2)} = \max_{1\le i\le m_2}\left\{f_{2i}(y^{(1)}_{n-k^{(2)}_{i,1}}, y^{(2)}_{n-k^{(2)}_{i,2}}, \ldots, y^{(l)}_{n-k^{(2)}_{i,l}}, n), y^{(2)}_{n-s}\right\}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ y_n^{(l)} = \max_{1\le i\le m_l}\left\{f_{li}(y^{(1)}_{n-k^{(l)}_{i,1}}, y^{(2)}_{n-k^{(l)}_{i,2}}, \ldots, y^{(l)}_{n-k^{(l)}_{i,l}}, n), y^{(l)}_{n-s}\right\}, \end{cases} \quad n\in\mathbb{N}_0 \tag{1.2}$$

where $s, l, m_j, k^{(j)}_{i,t} \in \mathbb{N} \equiv \{1,2,\ldots\}$ $(j,t\in\{1,2,\ldots,l\})$ and $f_{ji} : \mathbb{R}^l_+ \times \mathbb{N}_0 \longrightarrow \mathbb{R}_+$ ($j\in\{1,\ldots,l\}$ and $i\in\{1,\ldots,m_j\}$), and showed that every positive solution of (1.2) is eventually periodic with (not necessarily prime) period s if f_{ji} satisfy some conditions.

Moreover, Stević et al. [28] investigated the following max-type system of difference equations

$$\begin{cases} y_n^{(1)} = \max_{1\le i_1\le m_1}\left\{f_{1i_1}(y^{(1)}_{n-k^{(1)}_{i_1,1}}, y^{(2)}_{n-k^{(1)}_{i_1,2}}, \ldots, y^{(l)}_{n-k^{(1)}_{i_1,l}}, n), y^{(\sigma(1))}_{n-t_1 s}\right\}, \\ x_n^{(2)} = \max_{1\le i_2\le m_2}\left\{f_{2i_2}(y^{(1)}_{n-k^{(2)}_{i_2,1}}, y^{(2)}_{n-k^{(2)}_{i_2,2}}, \ldots, y^{(l)}_{n-k^{(2)}_{i_2,l}}, n), y^{(\sigma(2))}_{n-t_2 s}\right\}, \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ y_n^{(l)} = \max_{1\le i_l\le m_l}\left\{f_{li_l}(y^{(1)}_{n-k^{(l)}_{i_l,1}}, y^{(2)}_{n-k^{(l)}_{i_l,2}}, \ldots, y^{(l)}_{n-k^{(l)}_{i_l,l}}, n), y^{(\sigma(l))}_{n-t_l s}\right\}, \end{cases} \quad n\in\mathbb{N}_0 \tag{1.3}$$

where $s, l, m_j, t_j, k^{(j)}_{i_j,h} \in \mathbb{N}$ $(j,h\in\{1,2,\ldots,l\})$, $(\sigma(1),\ldots,\sigma(l))$ is a permutation of $(1,\ldots,l)$ and $f_{ji_j} : \mathbb{R}^l_+ \times \mathbb{N}_0 \longrightarrow \mathbb{R}_+$ ($j\in\{1,\ldots,l\}$ and $i_j\in\{1,\ldots,m_j\}$). They showed that every positive solution of (1.3) is eventually periodic with period sT for some $T\in\mathbb{N}$ if f_{ji_j} satisfy some conditions.

In 2015, Yazlik et al. [35] studied the following max-type system of difference equations

$$\begin{cases} x_{n+1} = \max\left\{\dfrac{1}{x_n}, \min\left\{1, \dfrac{\alpha}{y_n}\right\}\right\}, \\ y_{n+1} = \max\left\{\dfrac{1}{y_n}, \min\left\{1, \dfrac{\alpha}{x_n}\right\}\right\}, \end{cases} \quad n\in\mathbb{N}_0 \tag{1.4}$$

where $\alpha\in\mathbb{R}_+$ and the initial values $x_0, y_0\in\mathbb{R}_+$, and obtained in an elegant way the general solution of (1.4).

Motivated by aforementioned papers, in this paper, we study solutions of max-type system of difference equations

$$\begin{cases} x_n = \max\left\{\dfrac{1}{x_{n-m}}, \min\left\{1, \dfrac{A}{y_{n-r}}\right\}\right\}, \\ y_n = \max\left\{\dfrac{1}{y_{n-m}}, \min\left\{1, \dfrac{B}{x_{n-t}}\right\}\right\}, \end{cases} \quad n\in\mathbb{N}_0 \tag{1.5}$$

where $A, B\in\mathbb{R}_+$, $m, r, t\in\mathbb{N}$ with $r\neq m$ and $t\neq m$ and the initial values $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \ldots, x_{-1}, y_{-1}\in\mathbb{R}_+$ with $d = \max\{m, r, t\}$. We will show that every solution of (1.5) is eventually periodic with period $2m$.

2. Main results and proofs

In this section, we study the eventual periodicity of solutions of system (1.5). Let $\{(x_n, y_n)\}_{n\ge -d}$ be a solution of (1.5) with the initial values $x_{-d}, y_{-d}, x_{-d+1}, y_{-d+1}, \ldots, x_{-1}, y_{-1}\in\mathbb{R}_+$. Write

$$p_n = \min\left\{1, \frac{A}{y_{n-r}}\right\}, \qquad q_n = \min\left\{1, \frac{B}{x_{n-t}}\right\}.$$

Then we have $p_n \le 1$ and $q_n \le 1$ for any $n\in\mathbb{N}_0$. For the sake of easier presentation, we formulate and prove the following lemmas.

Lemma 2.1. *The following statements hold.*

(1) $x_n x_{n-m} \ge 1$ *(resp.* $y_n y_{n-m} \ge 1$ *) for all* $n\in\mathbb{N}_0$.
(2) $x_n \le \max\{x_{n-2m}, p_n\}$ *(resp.* $y_n \le \max\{y_{n-2m}, q_n\}$*) for all* $n\ge d$.
(3) *If* $x_n = 1/x_{n-m}$ *(resp.* $y_n = 1/y_{n-m}$*) for some* $n\ge d$*, then* $x_n \le x_{n-2m}$*(resp.* $y_n \le y_{n-2m}$*). If* $x_n = p_n > 1/x_{n-m}$ *(resp.* $y_n = q_n > 1/y_{n-m}$*) for some* $n\ge d$*, then* $x_n > x_{n-2m}$*(resp.* $y_n > y_{n-2m}$*).*

Proof.

(1) Since $x_n \ge 1/x_{n-m}$ (resp. $y_n \ge 1/y_{n-m}$) for all $n\in\mathbb{N}_0$, it follows $x_n x_{n-m} \ge 1$ (resp. $y_n y_{n-m} \ge 1$).