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# Infinitely many homoclinic solutions for a class of subquadratic second-order Hamiltonian systems

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ABSTRACT

In this paper, we mainly consider the existence of infinitely many homoclinic solutions for a class of subquadratic second-order Hamiltonian systems $\ddot{u} - L(t)u + W_u(t, u) = 0$, where $L(t)$ is not necessarily positive definite and the growth rate of potential function $W$ can be in (1, 3/2). Using the variant fountain theorem, we obtain the existence of infinitely many homoclinic solutions for the second-order Hamiltonian systems.



## 1. Introduction and main results

The aim of this paper is to study the following second-order Hamiltonian systems

$$\ddot{u} - L(t)u + W_u(t, u) = 0, \quad \forall\, t \in \mathbb{R} \tag{HS}$$

where $u = (u_1, u_2, \ldots, u_N) \in \mathbb{R}^N$, $W \in C^1(\mathbb{R} \times \mathbb{R}^N, \mathbb{R})$ and $L \in C(\mathbb{R}, \mathbb{R}^{N\times N})$ is a symmetric matrix-valued function. We usually say that a solution $u$ of (HS) is homoclinic (to 0) if $u \in C^2(\mathbb{R}, \mathbb{R}^N)$, $u(t) \to 0$ and $\dot{u}(t) \to 0$ as $t \to \pm\infty$. Furthermore, if $u \not\equiv 0$, then $u$ is called nontrivial.

In the applied sciences, Hamiltonian systems can be used in many practical problems regarding gas dynamics, fluid mechanics and celestial mechanics. It is clear that the existence of homoclinic solutions is one of the most important problems in the theory of Hamiltonian systems. Recently, more and more mathematicians have paid their attention to the existence and multiplicity of homoclinic orbits for Hamiltonian systems, see [1–21].

For the case of that $L(t)$ and $W(t, x)$ are either independent of $t$ or periodic in $t$, there have been several excellent results, see [1–3,7,8,12–16]. More precisely, in the paper [16], Rabinowitz has proved the existence of homoclinic orbits as a limit of $2kT$-periodic solutions of (HS). Later, using the same method, several results for general Hamiltonian systems were obtained by Izydorek and Janczewska [8], Lv et al. [12].

If $L(t)$ and $W(t, x)$ are not periodic with respect to $t$, it will become more difficult to consider the existence of homoclinic orbits for (HS). This problem is quite different from the case mentioned above, due to the lack of compactness of the Sobolev embedding. In [17], Rabinowitz and Tanaka investigated system (HS) without periodicity, both for $L$ and $W$. Specifically, they assumed that the smallest eigenvalue of $L(t)$ tends to $+\infty$ as $|t| \to \infty$, and showed that system (HS) admits a homoclinic orbit by using a variant of the Mountain Pass theorem without the Palais–Smale condition. Inspired by the work of Rabinowitz and Tanaka [17], many results [4,6,10,11,14,15,18,20,21] were obtained for the case of aperiodicity. Most of them were

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presented under the following condition that $L(t)$ is positive definite for all $t \in \mathbb{R}$,

$$(L(t)u, u) > 0, \quad \forall\ t \in \mathbb{R} \text{ and } u \in \mathbb{R}^N \backslash \{0\}.$$

Motivated by [6,20], in this article we will study the existence of infinitely many homoclinic solutions for (HS), where $L(t)$ is not necessarily positive definite for all $t \in \mathbb{R}$ and the growth rate of potential function $W$ can be in (1, 3/2). The main tool is the variant fountain theorem established in [22]. Our main results are the following theorems.

**Theorem 1.1.** *Assume that L and W satisfy the following conditions:*

(L1) *There exists an $\alpha < 1$ such that*

$$l(t)|t|^{\alpha-2} \to \infty \quad as\ |t| \to \infty$$

*where* $l(t) := \inf_{|u|=1, u \in \mathbb{R}^N} (L(t)u, u)$ *is the smallest eigenvalue of* $L(t)$*;*

(L2) *There exist constants $\bar{a} > 0$ and $\bar{r} > 0$ such that*

(i) $L \in C^1(\mathbb{R}, \mathbb{R}^{N\times N})$ *and* $|L'(t)u| \le \bar{a}|L(t)u|$, $\forall\ |t| > \bar{r}$ *and* $u \in \mathbb{R}^N$ *, or*

(ii) $L \in C^2(\mathbb{R}, \mathbb{R}^{N\times N})$ *and* $\big((L''(t) - \bar{a}L(t))u, u\big) \le 0$, $\forall\ |t| > \bar{r}$ *and* $u \in \mathbb{R}^N$ *,*

*where* $L'(t) = (d/dt)L(t)$ *and* $L''(t) = (d^2/dt^2)L(t)$*; (W)* $W(t,u) = a(t)|u|^\nu$ *where* $a : \mathbb{R} \to \mathbb{R}^+$ *is a continuous function such that* $a \in L^\mu(\mathbb{R}, \mathbb{R})$, $1 < \nu < 2$ *is a constant,* $2 \le \mu \le \bar{\nu}$ *and*

$$\bar{\nu} = \begin{cases} \dfrac{2}{3-2\nu}, & 1 < \nu < \dfrac{3}{2} \\ \infty, & \dfrac{3}{2} \le \nu < 2 \end{cases}$$

*Then (HS) possesses infinitely many homoclinic solutions.*

**Remark 1.2.** When we choose $\nu \in (1, \frac{3}{2})$, it is easy to see that $W$ satisfies the condition (W) of Theorem 1.1 but does not satisfy the corresponding conditions in [6,20]. Furthermore, the constant $\mu$ can be change in $[2, \bar{\nu}]$.

## 2. Preliminaries

In this section, for the purpose of readability and making this paper self-contained, we will show the variational setting for (HS), which can be found in [6,20]. In what follows, we will always assume that $L(t)$ satisfies (L1). Let $\mathcal{A}$ be the selfadjoint extension of the operator $-(d^2/dt^2) + L(t)$ with domain $\mathscr{D}(\mathcal{A}) \subset L^2 \equiv L^2(\mathbb{R}, \mathbb{R}^N)$. Let us write $\{E(\lambda) : -\infty < \lambda < +\infty\}$ and $|\mathcal{A}|$ for the spectral resolution and the absolute value of $\mathcal{A}$ respectively, and denote by $|\mathcal{A}|^{1/2}$ the square root of $|\mathcal{A}|$. Define $U = I - E(0) - E(-0)$. Then $U$ commutes with $\mathcal{A}$, $|\mathcal{A}|$ and $|\mathcal{A}|^{1/2}$, and $\mathcal{A} = U|\mathcal{A}|$ is the polar decomposition of $\mathcal{A}$ (see [9]). We write $E = \mathscr{D}(|\mathcal{A}|^{1/2})$ and introduce the following inner product on $E$

$$(u, v)_0 = (|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v)_2 + (u, v)_2$$

and norm

$$\|u\|_0 = (u, u)_0^{1/2}.$$

Here, $(\,\cdot, \cdot)_2$ denotes the usual $L^2$-inner product. Therefore, $E$ is a Hilbert space. Since $C_0^\infty(\mathbb{R}, \mathbb{R}^N)$ is dense in $E$, it is obvious that $E$ is continuous embedded in $H^1(\mathbb{R}, \mathbb{R}^N)$ (see [6]). Furthermore, we have the following lemmas by [6].

**Lemma 2.1.** *If L satisfies (L1), then E is compactly embedded in* $L^p \equiv L^p(\mathbb{R}, \mathbb{R}^N)$ *for all* $1 \le p \in (2/(3-\alpha), \infty]$.

**Lemma 2.2.** *Let L satisfies (L1) and (L2), then* $\mathscr{D}(\mathcal{A})$ *is continuously embedded in* $W^{2,2}(\mathbb{R}, \mathbb{R}^N)$*, and consequently, we have*

$$|u(t)| \to 0 \quad and \quad |\dot{u}(t)| \to 0 \quad as\ |t| \to \infty, \quad \forall\ u \in \mathscr{D}(\mathcal{A}).$$

From [6], combining (L1) and Lemma 2.1, we can prove that $\mathcal{A}$ possesses a compact resolvent. Consequently, the spectrum $\sigma(\mathcal{A})$ consists of eigenvalues, which can be arranged as $\lambda_1 \le \lambda_2 \le \cdots \to \infty$ (counted with multiplicity), and the corresponding system of eigenfunctions $\{e_n : n \in \mathbb{N}\}$, $\mathcal{A}e_n = \lambda_n e_n$, which forms an orthogonal basis in $L^2$. Next, we define

$$n^- = \#\{i|\lambda_i < 0\}, \ n^0 = \#\{i|\lambda_i = 0\}, \ \bar{n} = n^- + n^0$$

and

$$E^- = \text{span}\{e_1, \ldots, e_{n^-}\}, \ E^0 = \text{span}\{e_{n^-+1}, \ldots, e_{\bar{n}}\} = \text{Ker}\mathcal{A}, \ E^+ = \overline{\text{span}\{e_{\bar{n}+1}, \ldots\}}.$$

Here, the closure is taken in $E$ with respect to the norm $\|\cdot\|_0$. Then

$$E = E^- \oplus E^0 \oplus E^+.$$

Furthermore, we define on $E$ the following inner product

$$(u, v) = (|\mathcal{A}|^{1/2}u, |\mathcal{A}|^{1/2}v)_2 + (u^0, v^0)_2,$$