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Newton method for symmetric quartic polynomial

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A R T I C L E I N F O

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A B S T R A C T

We investigate the parameter plane of the Newton's method applied to the family of quartic polynomials $p_{a,b}(z)=z^4+az^3+bz^2+az+1$, where a and b are real parameters. We divide the parameter plane $(a,b)\in\mathbb{R}^2$ into twelve open and connected *regions* where p, p' and p'' have simple roots. In each of these regions we focus on the study of the Newton's operator acting on the Riemann sphere.

1. Introduction

Newton's method is the universal root finding algorithm in all scientific areas of knowledge. It is also the seed of what we know as holomorphic dynamics and it goes back to Ernest Schröder and Artur Caley who investigated the global dynamics of Newton's method applied to low degree polynomials as a rational map defined on the Riemann sphere. This global study is not only theoretical but it also has important implications at computational level (see for instance [6]).

Given a rational map $f:\hat{\mathbb{C}}\to\hat{\mathbb{C}}$, where $\hat{\mathbb{C}}$ denotes the Riemann sphere, we consider the dynamical system given by the iterates of f. The Riemann sphere splits into two totally $f-$invariant subsets: the *Fatou set* $\mathcal{F}(f)$, which is defined to be the set of points $z\in\hat{\mathbb{C}}$ where the family $\{f^n, n\geq 1\}$ is normal in some neighborhood of z, and its complement, the *Julia set* $\mathcal{J}(f)=\hat{\mathbb{C}}\setminus\mathcal{F}(f)$. The Fatou set is open and therefore $\mathcal{J}(f)$ is closed. Moreover, if the degree of the rational map f is greater than or equal to 2, then the Julia set $\mathcal{J}(f)$ is not empty and it is the closure of the set of repelling periodic points of f.

The connected components of $\mathcal{F}(f)$, called *Fatou components*, are mapped under f among themselves. If follows from the Classification Theorem ([9], Theorem 13.1) that any periodic Fatou component of a rational map is either the basin of attraction of an attracting or parabolic cycle or a simply connected rotation domain (a Siegel disk) or a doubly connected component rotation domain (a Herman ring). Moreover, the basin of attraction of an attracting or parabolic cycle contains, at least, one critical point i.e. a point $z\in\hat{\mathbb{C}}$ such that $f'(z)=0$. For a background on the dynamics of rational maps we refer to [1,5,9].

Given p a polynomial of degree $d\geq 2$ we define the Newton's map as

$$N_p(z):=z-\frac{p(z)}{p'(z)}.$$

Clearly, roots of p correspond to attracting fixed points of N_p. It is well-known (see [11]) that $\mathcal{J}(N_p)$ is connected (see also [2,3]) and consequently, all Fatou components are simply connected. Although, as we claimed, Newton's method is *the*

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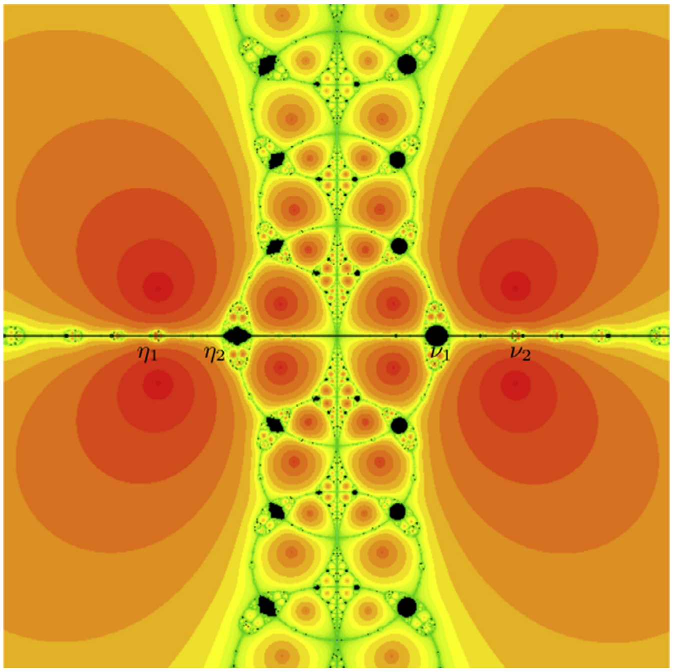

Fig. 1. Dynamical plane of $N_{a,\,b}$ for $a = 0.0013$ and $b = -1.73729$ in the region $\mathcal{R}_4$. We can see in red the four basins of attractions of the complex roots of $p_{a,\,b}$ and in black the two attracting cycles of period two, $\{\eta_1,\ \eta_2\}$ and $\{\nu_1,\ \nu_2\}$, to which the *free* critical points are attracted. (For interpretation of the references to color in this figure legend, the reader is referred to the web version of this article.)

universal root finding algorithm it reveals limitations. Precisely, for some polynomials of degree $d \geq 3$, there are open sets of initial conditions in the dynamical plane not converging to any root of p. The reason for this is the existence of at least one *free* critical point which allows the Newton's map of p (for an open set of polynomials in parameter space) to have attracting basins of period $k \geq 2$. Of course for all seeds on those basins the iterates are not converging to any of the roots of p. See [7] for a remarkable discussion in this direction.

There are also several results about the dynamical plane as well as the parameter plane of Newton's method applied to some concrete families of polynomials. The most studied case is Newton's method of cubic polynomials $q(z) = z(z-1)(z-a)$, $a \in \mathbb{C}$, for which the only *free* critical point is located at $(a+1)/3$. See [10,12] and references therein.

However, there is not a general study on Newton's method for quartic polynomials. Since the degree of N_p in this case is $d = 4$ (we assume no double roots of p to keep inside the family), we know there are $2d - 2 = 6$ critical points. Of course four of them correspond to the four roots of p but we have two *free* extra critical points and hence the parameter space is $\mathbb{C}^2$. In our approach we decrease the dimension of the parameter space but somehow we keep the difficulty. Indeed, we consider the family of symmetric quartic polynomials with two real parameters. Thus, although the parameter space is $\mathbb{R}^2$, the number of *free* critical points is still two. Fig. 1 illustrates one case where the two critical points are attracted by two different period two attracting cycles.

Precisely, the main goal of this paper is to study some topological properties of the parameter and dynamical plane of Newton's method applied to the family of four degree symmetric polynomials:

$$p_{a,b} := p_{a,b}(z) = z^4 + az^3 + bz^2 + az + 1, \tag{1}$$

when a and b are real parameters. Symmetric polynomials frequently appear along the dynamical study of others families of iterative methods (see [4], for example). We will split the parameter plane into regions in which the roots of the polynomials p, p' and p'' have simple zeroes and determine in which of those parameter regions we can guarantee that, except for a measure zero set (which is no relevant from the numerical point of view), any seed in dynamical plane converges to a root of p (Propositions 1 and 2). We also give numerical evidences that in other regions a more complicated and chaotic dynamics is possible. From the theoretical point of view these results are a first step in the direction of having a better understanding of Newton's method applied to quartic polynomials.

The expression of the Newton's map applied to $p_{a,\,b}$ writes as

$$N := N_{a,b}(z) = z - \frac{p_{a,b}(z)}{p'_{a,b}(z)} = z - \frac{z^4 + az^3 + bz^2 + az + 1}{4z^3 + 3az^2 + 2bz + a}. \tag{2}$$

The critical points of N are the solutions of $N'(z) = 0$; that is, the roots of p and p''. For each root $r_i(a, b) := r_i$, $i = 1, \ldots, 4$ we define its *basin of attraction*, $\mathcal{A}_{a,b}(r_i)$, as the set of points in the complex plane which tend to r_i under the Newton's map