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Error estimates for approximation of coupled best proximity points for cyclic contractive maps

A. Ilchev, B. Zlatanov*

Faculty of Mathematics and Informatics, Plovdiv University "Paisii Hilendarski", 24 "Tzar Assen" Str., Plovdiv 4000, Bulgaria

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ABSTRACT

We enrich the known results about coupled fixed and best proximity points of cyclic contraction ordered pair of maps. The uniqueness of the coupled best proximity points for cyclic contraction ordered pair of maps in a uniformly convex Banach space is proven. We find a priori and a posteriori error estimates for the coupled best proximity points, obtained by sequences of successive iterations, when the underlying Banach space has modulus of convexity of power type. A looser conditions are presented for the existence and uniqueness of coupled fixed points of a cyclic contraction ordered pair of maps in a complete metric space and a priori, a posteriori error estimates and the rate of convergence for the coupled fixed points are obtained for the sequences of successive iterations. We apply these results for solving systems of integral equations, systems of linear and nonlinear equations.

1. Introduction

In modeling real world processes, ordinary differential equations, integral equations and integrodifferential equations play a crucial role in many areas such as natural sciences, applied mathematics and engineering. One approach for solving the above mentioned problems is the Banach Contraction Principle, which is the fundamental result in fixed point theory. The theory of fixed points is an important tool for solving equations $Tx = x$ for mappings T defined on subsets of metric spaces or normed spaces. One kind of generalization of the Banach Contraction Principle is the notion of cyclical maps [1], i.e. $T(A) \subseteq B$ and $T(B) \subseteq A$. Since a non-self mapping $T: A \to B$ does not necessarily have a fixed point, one often attempts to find an element x which is in some sense closest to Tx. Best proximity point theorems are relevant in this perspective. The notion of best proximity point is introduced in [2]. This definition is more general than the concept of cyclic maps, in sense that if the sets intersect, then every best proximity point is a fixed point. A sufficient condition for existence and the uniqueness of best proximity points in uniformly convex Banach spaces is given in [2]. Since publication [2] the problem for existence and uniqueness of best proximity point has been widely investigated and the research on this problem continues.

The well known best approximation theorem of Fan [3] has been of great importance in nonlinear analysis, approximation theory, minimax theory, game theory, fixed point theory and variational inequalities. The concept of coupled fixed point theorem is introduced in [4]. Later on Bhaskar and Lakshmikantham [5] introduced the notions of a mixed monotone mapping, studied the problems of uniqueness of a coupled fixed point in partially ordered metric spaces and applied their theorems to problems of the existence of solution for a periodic boundary value problem. Harjani et al. obtained in [6] some

* Corresponding author. Tel.: +359 898477827.
E-mail addresses: atanasilchev1@gmail.com (A. Ilchev), bzlatanov@gmail.com (B. Zlatanov).

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coupled fixed point theorems for a mixed monotone operator in a complete metric space endowed with a partial order by using altering distance functions. They applied their results to the study of the existence and uniqueness of a nonlinear integral equation. Recent results about the application of coupled fixed point theorems for solving integral equations are obtained in [7].

There are many problems about fixed points and best proximity points that are not easy to be solved or could not be solved exactly. One of the advantages of Banach fixed point theorem is the error estimates of the successive iterations and the rate of convergence. That is why an estimation of the error when an iterative process is used is of interest. An extensive study about approximations of fixed points can be found in [8]. Very recently numerical solutions of fuzzy deferential equations [9] and a powerful kernel algorithm for solving fuzzy Fredholm–Volterra integrodifferential equations [10] have been presented.

A first result in the approximation of the sequence of successive iterations, which converges to the best proximity point for cyclic contractions is obtained in [11]. We have tried to expand the above result for coupled best proximity point. We have found "a priori error estimates" and "a posteriori error estimates" for the coupled best proximity point and for the coupled fixed point, which are obtained through a sequence of successive iterations in [12,13]. The advantage of the presented results is that a direct stop criteria of the iteration process is presented, when an exact solution is not possible to be found. The second benefit of the presented technique is that it widens the classes of equations for which an approximation of the solution can be found with sequences of successive iterations.

We illustrate the results by finding an estimate of the error for the examples from [12,13]. We present an example of system of integral equations and a system of nonlinear equations that can be solved with the technique of coupled best proximity points.

2. Preliminaries

In this section we give some basic definitions and concepts which are useful and related to the best proximity points. Let (X, ρ) be a metric space. Define a distance between two subsets $A, B \subset X$ by $\mathrm{dist}(A, B) = \inf\{\rho(x, y) : x \in A, y \in B\}$. For simplicity of the notations we will denote $\mathrm{dist}(A, B)$ with d.

Let A and B be nonempty subsets of a metric space (X, ρ). The map $T : A \bigcup B \to A \bigcup B$ is called a cyclic map if $T(A) \subseteq B$ and $T(B) \subseteq A$. A point $\xi \in A$ is called a best proximity point of the cyclic map T in A if $\rho(\xi, T\xi) = \mathrm{dist}(A, B)$.

Let A and B be nonempty subsets of a metric space (X, ρ). The map $T : A \bigcup B \to A \bigcup B$ is called a cyclic contraction map if T is a cyclic map and for some $k \in (0, 1)$ there holds the inequality $\rho(Tx, Ty) \le k\rho(x, y) + (1 - k)d$ for any $x \in A$, $y \in B$. The definition for cyclic contraction is introduced in [2].

The concept of coupled best proximity point theorem is introduced in [12].

Definition 1. [12] Let A and B be nonempty subsets of a metric space X, $F: A \times A \to B$. An ordered pair $(x, y) \in A \times A$ is called a coupled best proximity point of F if

$$\rho(x, F(x, y)) = \rho(y, F(y, x)) = d.$$

Definition 2. [4] Let A and B be nonempty subsets of a metric space X, $F: A \times A \to A$. An ordered pair $(x, y) \in A \times A$ is said to be a coupled fixed point of F in A if $x = F(x, y)$ and $y = F(y, x)$.

It is easy to see that if $A = B$ in Definition 1, then a coupled best proximity point reduces to a coupled fixed point.

Definition 3. [12] Let A and B be nonempty subsets of a metric space X. Let $F: A \times A \to B$ and $G: B \times B \to A$. For any pair $(x, y) \in A \times A$ we define the sequences $\{x_n\}_{n=0}^{\infty}$ and $\{y_n\}_{n=0}^{\infty}$ by $x_0 = x$, $y_0 = y$ and

$$x_{2n+1} = F(x_{2n}, y_{2n}), \quad y_{2n+1} = F(y_{2n}, x_{2n})$$
$$x_{2n+2} = G(x_{2n+1}, y_{2n+1}), \; y_{2n+2} = G(y_{2n+1}, x_{2n+1})$$

for all $n \ge 0$.

The notion of a cyclic contraction for two mappings, which generalizes the notion of cyclic contraction [2], is introduced in [12].

Definition 4. [12,13] Let A and B be nonempty subsets of a metric space X, $F: A \times A \to B$ and $G: B \times B \to A$. The ordered pair (F, G) is said to be a cyclic contraction if there exist non-negative numbers α, β, such that $\alpha + \beta < 1$ and there holds the inequality

$$\rho(F(x, y), G(u, v)) \le \alpha\rho(x, u) + \beta\rho(y, v) + (1 - (\alpha + \beta))d(A, B)$$

for all $(x, y) \in A \times A$ and $(u, v) \in B \times B$.

If $\alpha = \beta$ in Definition 4, we get the maps that are investigated in [12].

The best proximity results need norm-structure of the space X.

When we investigate a Banach space $(X, \|\cdot\|)$, we will always consider the distance between the elements to be generated by the norm $\|\cdot\|$ i.e. $\rho(x, y) = \|x - y\|$. We will denote the unit sphere and the unit ball of a Banach space $(X, \|\cdot\|)$ by S_X and B_X, respectively.