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Moving mesh discontinuous Galerkin methods for PDEs with traveling waves

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ABSTRACT

In this paper, a moving mesh discontinuous Galerkin (dG) method is developed for nonlinear partial differential equations (PDEs) with traveling wave solutions. The moving mesh strategy for one dimensional PDEs is based on the rezoning approach which decouples the solution of the PDE from the moving mesh equation. We show that the dG moving mesh method is able to resolve sharp wave fronts and wave speeds accurately for the optimal, arc-length and curvature monitor functions. Numerical results reveal the efficiency of the proposed moving mesh dG method for solving Burgers', Burgers'–Fisher and Schlögl (Nagumo) equations.

1. Introduction

The discontinuous Galerkin (dG) method is one of the most powerful discretization techniques for solving partial differential equations (PDEs) [1,2], especially for convection dominated problems, exhibiting localized phenomena like sharp traveling wave fronts, internal and boundary layers [3,4]. The dG method has been applied for this kind of singularly perturbed linear and nonlinear PDEs extensively using h-adaptive (refinement and coarsening in space), p-adaptive (enrichment of the local polynomial degree), hp-adaptive and space-time adaptive methods in the last two decades. Another approach to deal with these kind of problems, is the r-method or moving mesh method. In the moving mesh method the grid points are relocated in the regions where the solution shows rapid variation, while keeping the number of the nodes fixed. The dG discretization is very flexible, since there is no continuity requirement between the inter-element boundaries, which makes it suitable as a moving mesh method on irregular meshes. Most of the studies with moving mesh methods are limited to finite difference and continuous finite element discretization [5]. There are only few publications dealing with dG moving mesh method. They include the interior penalty dG method for preprocessing the solutions of steady state diffusion–convection–reaction equations [6], and the local dG moving mesh method for hyperbolic conservation laws [7].

In this paper we develop an adaptive dG moving mesh method for one dimensional semi-linear differential equations with traveling wave solutions of the form

$$u_t = \epsilon u_{xx} - f(u, u_x), \quad (x,t) \in \Omega \times (t_0, T_f] \tag{1a}$$

$$u(x_L, t) = u_L, \quad u(x_R, t) = u_R, \quad t \in (t_0, T_f] \tag{1b}$$

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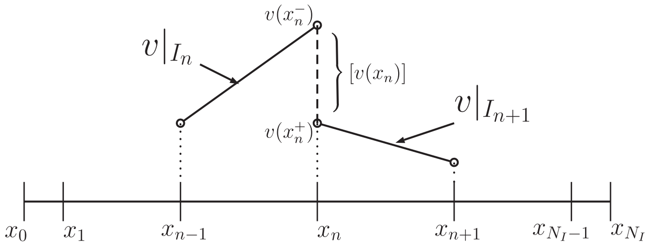

Fig. 1. Jump and limit terms of a piecewise discontinuous function $v(x)$.

$$u(x, t_0) = u_0, \quad x \in \Omega, \tag{1c}$$

where $\Omega = [x_L, x_R] \subset \mathbb{R}$, t_0 and T_f are the initial and final time instances, respectively, and ϵ denotes the diffusion coefficient. The model Eq. (1) becomes the Burgers equation with $f(u, u_x) = uu_x$ [8], Burgers'–Fisher equation with $f(u, u_x) = \alpha uu_x + \beta u(u-1)$ [8] and the Schlögl or Nagumo equation with $f(u, u_x) = u(1-u)(1-\beta)/\delta$ [5].

A moving mesh method has three main components; the discretization of the physical PDE, mesh strategy using monitor functions and discretization of the mesh equation. The discretization of physical PDE is either coupled with the moving mesh equation or separated. In the quasi-Lagrangian approach, a large system of the discretized PDE and moving mesh equation are solved simultaneously by the standard ordinary differential equation (ODE) solvers. Instead, we use the rezoning approach by solving alternately the PDE and mesh equation, which allows more flexibility; mesh generation can be coded separately and embedded in the solution of the PDE. Since the mesh is updated at each time step, the physical PDE has to be discretized at the next time step on the new mesh. We use the static rezoning approach with the same number of points at each time step [9] in contrast to the dynamic rezoning [10] where the number of mesh points is changed at every time step. Therefore in the static rezoning approach the solutions from old to new mesh have to be interpolated.

The paper is organized as follows. In the next section we describe briefly the dG method for the 1D model problem (1) on a uniform fixed mesh. Moving mesh adaption strategy and the adaptive moving mesh dG algorithm is presented in Section 3. Numerical results are given in Section 4 to demonstrate the effectiveness of the proposed method.

2. Discretization of the problem on a fixed mesh

Before giving the moving mesh strategy in Section 3, in this section we describe the dG discretization of the model problem (1) on a fixed uniform mesh

$$\mathcal{T}_h: \quad x_n = x_L + nh, \quad n = 0, 1, \ldots, N_I, \tag{2}$$

consisting of N_I elements (sub-intervals) $I_n = [x_{n-1}, x_n]$, $n = 1, 2, \ldots, N_I$, and with the fixed mesh size $h = (x_R - x_L)/N_I$.

2.1. Space discretization by discontinuous Galerkin method

We use for the space discretization of the model problem (1) on a fixed mesh (2) the symmetric interior penalty Galerkin (SIPG) method [1,2] which is a member of the family of dG methods. The dG methods use the space of piecewise discontinuous polynomials of degree at most k:

$$V_h = \{v : v|_{I_n} \in \mathbb{P}_k(I_n), \ \forall n = 1, \ldots, N_I\},$$

where $\mathbb{P}_k(I_n)$ is the space of polynomials of degree at most k on an interval I_n. Since the functions in V_h are discontinuous at the inter-element nodes, we define the jump and average of a piecewise function v at the endpoints of I_n, $n = 1, \ldots, N_I - 1$, respectively, as depicted in Fig. 1,

$$[v(x_n)] = v(x_n^-) - v(x_n^+), \quad \{v(x_n)\} = \frac{1}{2}(v(x_n^-) + v(x_n^+)), \tag{3}$$

with

$$v(x_n^-) = \lim_{x \mapsto x_n^-} v(x), \quad v(x_n^+) = \lim_{x \mapsto x_n^+} v(x). \tag{4}$$

On the boundary nodes, the jump and average are defined as

$$[v(x_0)] = -v(x_0^+), \ \{v(x_0)\} = v(x_0^+), \ [v(x_{N_I})] = v(x_{N_I}^-), \ \{v(x_{N_I})\} = v(x_{N_I}^+). \tag{5}$$

The SIPG scheme is constructed by multiplying the continuous (the solution u is sufficiently smooth at the end points of I_n) Eq. (1) by a test function $v \in V_h$ and integrating by parts on each element I_n, $n = 1, \ldots, N_I$:

$$\int_{x_{n-1}}^{x_n} u_t v dx + \int_{x_{n-1}}^{x_n} \epsilon u_x v_x dx - \epsilon u_x(x_n) v(x_n^-) + \epsilon u_x(x_{n-1}) v(x_{n-1}^+) + \int_{x_{n-1}}^{x_n} f(u, u_x) v dx = 0.$$