Applied Mathematics and Computation 292 (2017) 47–56

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Optimal control of an epidemiological model with multiple time delays

Eihab B.M. Bashier[a], Kailash C. Patidar[b,1,*]

[a] Department of Applied Mathematics, Faculty of Mathematical Sciences, University of Khartoum, P.O. Box 321, Khartoum, Sudan
[b] Department of Mathematics and Applied Mathematics, University of the Western Cape, Private Bag X17, 7535 Bellville, South Africa

ARTICLE INFO

MSC:
34K18
49K25
65N06
65N12
90C30
92B99

Keywords:
Optimal control
Epidemiological models
Delay differential equations

ABSTRACT

In this paper, we consider an optimal control model governed by a system of delay differential equations representing an SIR model. We extend the model of Kaddar (2010) by incorporating the suitable controls. We consider two control strategies in the optimal control model, namely: the vaccination and treatment strategies. The model has three time delays that represent the incubation period, and the times taken by the vaccine and treatment to be effective. We derive the first-order necessary conditions for the optimal control and perform numerical simulations to show the effectiveness as well as the applicability of the model for different values of the time delays. These numerical simulations show that the model is more sensitive to the delays representing the incubation period and the treatment delay, whereas the delay associated with the vaccine is not significant.

1. Introduction

Epidemiological models are often used to describe the dynamics of epidemic diseases in populations. There have been many variations of such classical epidemiological models [13,14]. These models are based on the standard Susceptible-Infectious-Susceptible, Susceptible-Infectious-Recovered and Susceptible-Exposed-Infectious-Recovered models, which are determined according to the difference on the method of transmission, nature of the disease, e.g., those with short/long incubation period, killer/curable diseases, etc, and the response of the individuals to it, for instance, gaining transient/permanent immunity, dying from the disease, etc. [7,23,29]. The main purpose of formulating a such epidemiological model is to understand the long-term behavior of the epidemic disease and to determine the possible strategies to control it.

Differential equations, whether they are ordinary, delay, partial or stochastic, are one of the main mathematical tools being used to formulate many epidemiological models. The focus in such epidemiological models has been on the incidence rate at which people move from the class of susceptible individuals to the class of infective individuals. These incidence rates have been modeled mostly by using bilinear and Holling type of functional responses [12,15,20,27,28].

On the other hand, optimal control has extensively been used as a strategy to control the epidemic outbreaks [9]. The main idea behind using the optimal control in epidemics is to search for, among the available strategies, the most effective strategy that reduces the infection rate to a minimum level while optimizing the cost of deploying a therapy or a preventive

* Corresponding author. Fax: +27 219591241.
E-mail addresses: eihabbashier@gmail.com (E.B.M. Bashier), kpatidar@uwc.ac.za (K.C. Patidar).

1 The research of this author was also supported by South African National Research Foundation.

http://dx.doi.org/10.1016/j.amc.2016.07.009
0096-3003/

vaccine that is used for controlling the disease progression (see, for example, [30]). In terms of epidemic diseases, such strategies can include therapies, vaccines, isolation and educational campaigns [3,6,9,17].

Recently, many optimal control models pertaining to epidemic diseases have appeared in the literature. They include, but not limited to, delayed SIRS epidemic model [18] (where Laarbi et al. proposed an optimal control model governed by a delayed SIRS model with a single time delay representing the incubation period), delayed SIR model [1] (where Abta et al. formulated an optimal control model to optimize the costs of vaccination and treatment on a delayed SIR model with a single delay representing the incubation period and a saturated incidence rate), tuberculosis model [25] (where Silva and Torres proposed optimal control strategies to minimize the cost of interventions using data from Angola), HIV model [11] (where Hattaf and Yousfi used optimal controls to represent the efficiency of drug treatment in inhibiting viral production and preventing new infections), swine flu [5] (where Boklund et al. studied epidemiological and economic consequences of some control strategies in a classical swine flu epidemic under Danish conditions with respect to herd demographics and geography and investigated the effect of extra biosecurity measures on farms) and dengue fever [2] (where Aldila et al. designed an optimal control problem using four different control parameters and discussed some results about epidemic prevention and outbreak reduction strategies).

An important class of optimal control problems, is the class of problems characterized by delay differential equations. The existence theorems and the derivation of the first order optimality conditions for optimal control problem governed by delay differential equations have been discussed by many authors [4,8,10,19]. Barati [4] introduced a new approach based on embedding method for finding an approximate solution for a wide range of nonlinear optimal control problems with delays in state and control variables subject to mixed-control state constraints. In [8], Frankena considered a general optimal control problem involving ODEs with delayed arguments and a set of equality and inequality restrictions on state and control variables. Using variational techniques, she presented a maximum principle in pointwise form and then from this maximum principle, she derived necessary conditions for the existence of the optimal controls. Göllmann et al. [10] derived necessary optimality conditions in the form of Pontryagin's minimum principle for optimal control problems with delays in state and control variables. In [19], Nababan derived a Filippov-type lemma for functions involving delays. He then used this lemma to prove the existence of an optimal control for a class of nonlinear control processes with delays appearing in both state and control variables.

In this paper, we consider an optimal control problem governed by a system of delay differential equations with multiple time delays. The governing state equations of the optimal control model are described in a SIR framework with a saturated incidence rate and a time delay representing the incubation period. This model is found in [16]. Furthermore, we consider two additional time delays in the optimal control model. These delays represent the times taken by the vaccine and therapy before each of these two strategies become effective. Then we derive first-order necessary conditions for existence of the optimal control and develop a numerical method for solving them.

The rest of this paper is organized as follows. In Section 2, we give the statement of the optimal control problem. We derive the necessary conditions for existence of the optimal control in Section 3. In Section 4, we describe the numerical method and present the resulting numerical simulations. Finally, we discuss these results in Section 5 along with some concluding remarks.

2. Statement of the optimal control problem

Given the initial state of the population (S_0, I_0, R_0), our goal is to compute the optimal pair of vaccination and treatment strategies (u, v) that would maximize the recovered population and minimize both the infected and susceptible populations; and at the same time minimize the costs of applying the vaccination and treatment strategies. To achieve this goal, optimal control is an appropriate choice for expressing the above problem mathematically. Therefore, we consider in this paper an optimal control problem of the form

$$\min_{(u,v)\in\mathbb{U}} J(u(t), v(t)) = S(T) + I(T) - R(T) + \int_0^T \Big(C_1 u^2(t) + C_2 v^2(t) + S(t) + I(t) - R(t)\Big) dt, \tag{2.1}$$

subject to the state equations

$$\dot{S}(t) = \Omega - \frac{\beta S(t) I(t)}{1 + \alpha_1 S(t) + \alpha_2 I(t)} - \mu S(t) - u(t) S(t), \tag{2.2}$$

$$\dot{I}(t) = \frac{\beta S(t-\tau) I(t-\tau)}{1 + \alpha_1 S(t-\tau) + \alpha_2 I(t-\tau)} - (\mu + \alpha + \gamma) I(t) - v(t) I(t), \tag{2.3}$$

$$\dot{R}(t) = \gamma I(t) + u(t-\sigma) S(t-\sigma) + v(t-\delta) I(t-\delta) - \mu R(t), \tag{2.4}$$

and history data

$$S(\theta) = \varphi_S(\theta), \theta \le 0, \tag{2.5}$$