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The Jacobi and Gauss–Seidel-type iteration methods for the matrix equation $AXB = C$

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ABSTRACT

In this paper, the Jacobi and Gauss–Seidel-type iteration methods are proposed for solving the matrix equation $AXB = C$, which are based on the splitting schemes of the matrices A and B. The convergence and computational cost of these iteration methods are discussed. Furthermore, we give the preconditioned Jacobi and Gauss–Seidel-type iteration methods. Numerical examples are given to demonstrate the efficiency of these methods proposed in this paper.

1. Introduction

In this paper, we consider the solution of the matrix equation

$$AXB = C, \tag{1.1}$$

where A, B and $C \in R^{n \times n}$. The matrix Eq. (1.1) arises in several areas of applications such as signal and image processing [1], photogrammetry [3], etc.

Due to the importance of the matrix Eq. (1.1) in matrix theory and applications, hence how to effectively solve this kind of equation has been under research recently. When the matrices A and B are small and dense, direct methods such as QR-factorization-based algorithms [2,5] are attractive. However, these direct algorithms are quite costly and impractical when A and B are large. Therefore, iteration methods in [4,6,8,9,10,12,15,21] to solve the matrix Eq. (1.1) have attracted much interests recently. For example, the author in [4] presented an iteration method for the least squares symmetric solution of the matrix Eq. (1.1), which can obtain a solution within finite iteration steps in the absence of roundoff errors for any initial symmetric matrix, and the solution with least norm can be obtained by choosing a special kind of initial symmetric matrix. In [6,15], gradient-based iteration methods as well as least-squares-based iteration methods were used to solve the matrix Eq. (1.1). In [21], a Hermitian and skew-Hermitian splitting (HSS) iteration method is presented for solving the matrix Eq. (1.1), which is formed by extending the corresponding HSS iteration method [23] for solving $Ax = b$.

It is well known that the matrix Eq. (1.1) can be written the following mathematically equivalent matrix-vector form by Kronecker products symbol [7]

$$Tx = c,$$

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where $T = B^T \otimes A$, and the vectors x and c are the forms of vec(X) and vec(C). With the application of Kronecker products, many algorithms are proposed to solve the matrix Eq. (1.1) (see, e.g., [9,12,14,16,18,19,21]). In this paper, based on the matrix splittings of A and B, we present the Jacobi and Gauss–Seidel-type iteration methods. Furthermore, we give the preconditioned Jacobi and Gauss–Seidel-type iteration methods by preconditioning the matrices A or B, firstly.

This paper is organized as follows. In Section 2, we introduce some properties of the Kronecker products, several lemmas and theorems for iteration methods, which will be used in the following sections. We give the Jacobi and Gauss–Seidel-type iteration methods as well as their convergence properties in Section 3, and the corresponding algorithms are presented. In Section 4, the modified Jacobi and Gauss–Seidel-type iteration methods are presented, which can reduce the computational and storage cost greatly compared with the Jacobi and Gauss–Seidel-type iteration methods. Section 5 is devoted to the preconditioned Jacobi and Gauss–Seidel-type iteration methods, which converge much faster than the Jacobi and Gauss–Seidel-type iteration methods by choosing an appropriate preconditioner. In Section 6, several numerical examples are given to show the efficiency of the Jacobi and Gauss–Seidel-type iteration methods, the preconditioned Jacobi and Gauss–Seidel-type iteration methods, respectively. Finally, we conclude in Section 7.

2. Preliminaries

In this section, we will introduce some well-known definitions and results, which can be found in [7,11].

Let X be an $m \times n$ matrix, then vec(X) is defined to be a column vector of size $m \cdot n$ made of the columns of X stacked atop one another from left to right. Let A and B be $m \times n$ and $p \times q$ matrices, respectively. Then the Kronecker product $A \otimes B$ is the $(m \cdot p) \times (n \cdot q)$ matrix

$$\begin{bmatrix} a_{11} \cdot B & \cdots & a_{1n} \cdot B \\ \vdots & & \vdots \\ a_{m1} \cdot B & \cdots & a_{mn} \cdot B \end{bmatrix}.$$

Lemma 2.1. *Let A, B and X be $m \times m$, $n \times n$, and $m \times n$ matrices, respectively. Then the following properties hold:*

1. $\mathrm{vec}(AX) = (I_n \otimes A) \cdot \mathrm{vec}(X)$;
2. $\mathrm{vec}(XB) = (B^T \otimes I_m) \cdot \mathrm{vec}(X)$.

Lemma 2.2. *The following relations about Kronecker products hold:*

1. *Assume that the products $A \cdot C$ and $B \cdot D$ are well defined. Then $(A \otimes B) \cdot (C \otimes D) = (A \cdot C) \otimes (B \cdot D)$;*
2. *if A and B are invertible, then $(A \otimes B)^{-1} = A^{-1} \otimes B^{-1}$;*
3. $(A \otimes B)^T = A^T \otimes B^T$.

Lemma 2.3. *Let $\lambda(A)$ and $\mu(B)$ be the spectrums of $A \in R^{n \times n}$ and $B \in R^{m \times m}$, respectively, then*

$$\lambda(A \otimes B) = \{\lambda_i \mu_j : \lambda_i \in \lambda(A), \mu_j \in \mu(B), i = 1, 2, \ldots, n;\ j = 1, 2, \ldots, m\}.$$

Let $A = M - N$, then the pairs of matrices (M, N) of A is called a splitting of A if $\det(M) \neq 0$; a convergent splitting if $\rho(M^{-1}N) < 1$, where $\rho(C)$ denotes the spectral radius of the matrix C. The above splitting yields an iterative method as follows: $Ax = Mx - Nx = b$ implies $Mx = Nx + b$ or $x = M^{-1}Nx + M^{-1}b = Rx + c$. So we can take $x_{m+1} = Rx_m + c$ as our iteration sequence.

Lemma 2.4. *Let $\|\cdot\|$ be any operator norm ($\|R\| = \max_{x \neq 0} \frac{\|Rx\|}{\|x\|}$). If $\|R\| < 1$, then $x_{m+1} = Rx_m + c$ converges for any x_0.*

Lemma 2.5. *For all operator norms $\rho(R) \leq \|R\|$. For all R and for all $\varepsilon > 0$ there is an operator norm $\|R\|_\star \leq \rho(R) + \varepsilon$. The norm $\|\cdot\|_\star$ depends on both R and ε.*

Theorem 2.1. *The iteration $x_{m+1} = Rx_m + c$ converges to the solution of $Ax = b$ for all starting vectors x_0 and for all b if and only if $\rho(R) < 1$.*

3. The Jacobi and Gauss–Seidel-type iteration methods for the matrix Eq. (1.1)

By making use of Lemmas 2.1 and 2.2, the matrix Eq. (1.1) can be expressed as follows:

$$(B^T \otimes A)x = c, \tag{3.1}$$

where x and c are the forms of vec(X) and vec(C).

Let B^T be $\hat{B}$, then we have

$$(\hat{B} \otimes A)x = c. \tag{3.2}$$

We split $\hat{B}$ into the following form:

$$\hat{B} = \hat{M} - \hat{N}, \tag{3.3}$$

where $\hat{M}$ is nonsingular.