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A new multigrid finite element method for the transmission eigenvalue problems☆

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ABSTRACT

Numerical methods for the transmission eigenvalue problems are hot topics in recent years. Based on the work of Lin and Xie (2015), we build a multigrid method to solve the problems. With our method, we only need to solve a series of primal and dual eigenvalue problems on a coarse mesh and the associated boundary value problems on the finer and finer meshes. Theoretical analysis and numerical results show that our method is simple and easy to implement and is efficient for computing real and complex transmission eigenvalues.

1. Introduction

The transmission eigenvalue problems have theoretical importance in the uniqueness and reconstruction in inverse scattering theory [1,2]. Transmission eigenvalues can be determined from the far-field data of the scattered wave and used to obtain estimates for the material properties of the scattering object [3,4]. Many literatures such as [2,4–7] studied the existence of transmission eigenvalues, and authors in [4,8,9] explored the upper and lower bounds for the index of refraction $n(x)$.

In recent years, numerical methods for the transmission eigenvalue problems have attracted the attention from more and more researchers. The first numerical treatment of the transmission eigenvalue problem appeared in [10] where three finite element methods are proposed for the Helmholtz transmission eigenvalues. Later on, many other numerical methods were developed to solve the problems (see, e.g., [11–15]). In particular, Sun [11] proposed an iterative method and gave a coarse error analysis. Furthermore, Ji et al. [12] developed his work and proved the accurate error estimates for both eigenvalues and eigenfunctions by constructing an auxiliary problem as a bridge. An and Shen [13] proposed a spectral-element method to solve this problem numerically. Afterwards using the linearized technique the authors in [14,15] builded two new weak formulations and the corresponding finite element discretizations. Most of these works focus their attention on only a few lowest real transmission eigenvalues which are of practical importance in the scattering science to estimate the index of refraction [3].

The idea of multigrid methods for eigenvalue problems was developed originally from two grid methodology. In 2001, Xu and Zhou [16] proposed a two grid method based on inverse iteration for elliptic eigenvalue problems, which is, in a way, related to that in Lin and Xie [17]. After that, the two grid method was further developed into multigrid method and

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local and parallel algorithm in [18–21] et al. In recent years, Lin and Xie [22] and Xie [23] proposed a mutilevel correction method. This method enriches the finite element space at each correction step with the numerical eigenfunctions obtained from the last step. The aim of this enrich is to do the Aubin-Nitsche technique which is the important condition to do the two-grid correction. Such an enrich of the finite element space can significantly improve convergence rates and accuracy step by step, so that the multigrid method can achieve the same accuracy as solving the eigenvalue problem directly but with less computational work. Regarding the correction method of eigenvalue problems, some other related works can be referred to [24–26].

In this paper, based on the work in [22,23], we propose a new multigrid method to solve the transmission eigenvalue problems but based on the new weak formulation (2.6) proposed in [15] which is a linear and nonsymmetric eigenvalue problem. In this work, (1) we prove the error estimates of the transmission eigenvalues and eigenfunctions for our multigrid method. Our theoretical results are valid for arbitrary real and complex eigenvalues. (2) With our method, due to adopting the linearized weak formulation, we can transform the transmission eigenvalue problem into a generalized matrix eigenvalue problem and can be solved efficiently by the sparse solver *eigs* in Matlab; (3) with our multigrid method, the solution of eigenvalue problem on a fine mesh can be reduced to a series of the solutions of the eigenvalue problem on a coarse meshes and a series of solutions of the boundary value problems on the multilevel meshes. As numerical results indicate, this method is applicable to the real and complex transmission eigenvalues.

2. Preliminaries

Let $H^s(D)$ be a Sobolev space with norm $\|\cdot\|_s$ $(s=1,2)$, and

$$H_0^2(D)=\left\{v\in H^2(D): v|_{\partial D}=\frac{\partial v}{\partial \nu}|_{\partial D}=0\right\}.$$

Consider the Helmholtz transmission eigenvalue problem: Find $k\in\mathbb{C}$, ω, $\sigma\in L^2(D)$, $\omega-\sigma\in H^2(D)$ such that

$$\Delta\omega+k^2n(x)\omega=0,\ \ \text{in}\,D, \tag{2.1}$$

$$\Delta\sigma+k^2\sigma=0,\ \ \text{in}\,D, \tag{2.2}$$

$$\omega-\sigma=0,\ \ \text{on}\,\partial D, \tag{2.3}$$

$$\frac{\partial\omega}{\partial\nu}-\frac{\partial\sigma}{\partial\nu}=0,\ \ \text{on}\,\partial D, \tag{2.4}$$

where $D\subset\mathbb{R}^2$ or $D\subset\mathbb{R}^3$ is a bounded simply connected inhomogeneous medium, ν is the unit outward normal to ∂D.

It is possible to write (2.1)–(2.4) as an equivalent eigenvalue problem for $u=\omega-\sigma\in H_0^2(D)$. In particular,

$$(\Delta u+k^2u)=\Delta\omega+k^2\omega=k^2(1-n)\omega.$$

Dividing by $n-1$ and applying the operator $(\Delta+k^2n)$ to the above equality, the eigenvalue problem (2.1)–(2.4) can be stated as follows: Find $k^2\in\mathbb{C}$, $k^2\neq 0$, nonzero $u\in H_0^2(D)$ such that

$$\left(\frac{1}{n(x)-1}(\Delta u+k^2u),\,\Delta v+\bar{k}^2n(x)v\right)_0=0,\ \ \forall v\in H_0^2(D), \tag{2.5}$$

where $(\cdot,\cdot)_0$ is the inner product of $L^2(D)$. As usual, we define $\lambda=k^2$ as the transmission eigenvalue in this paper. We suppose that the index of refraction $n\in L^\infty(D)$ satisfying either one of the following assumptions

$$(C1)\ 1+\delta\le\inf_D n(x)\le n(x)\le\sup_D n(x)<\infty,$$

$$(C2)\ 0<\inf_D n(x)\le n(x)\le\sup_D n(x)<1-\varrho,$$

for some constant $\delta>0$ or $\varrho>0$.

For simplicity, in the coming discussion we assume $(C1)$ holds and $n(x)$ is proper smooth (for example $n(x)\in W^{2,\infty}(D)$). For the case $(C2)$ the argument method is the same.

Define Hilbert space $\mathbf{H}=H_0^2(D)\times L^2(D)$ with norm $\|(u,w)\|_{\mathbf{H}}=\|u\|_2+\|w\|_0$, and define $\mathbf{H}_s=H^s(D)\times H^{s-2}(D)$ with norm $\|(u,w)\|_{\mathbf{H}_s}=\|u\|_s+\|w\|_{s-2}$, $s=0,1$.

From (2.5) we derive that

$$\left(\frac{1}{n-1}\Delta u,\,\Delta v\right)_0-\lambda\left(\nabla\left(\frac{1}{n-1}u\right),\nabla v\right)_0-\lambda\left(\nabla u,\nabla\left(\frac{n}{n-1}v\right)\right)_0+\lambda^2\left(\frac{n}{n-1}u,v\right)_0=0,\ \ \forall v\in H_0^2(D).$$

Let $w=\lambda u$, we arrive at a linear weak formulation: Find $(\lambda,u,w)\in\mathbb{C}\times H_0^2(D)\times L^2(D)$ such that