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Anti-Ramsey numbers for matchings in 3-regular bipartite graphs

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ABSTRACT

The anti-Ramsey number $AR(K_n, H)$ was introduced by Erdős, Simonovits and Sós in 1973, which is defined to be the maximum number of colors in an edge coloring of the complete graph K_n without any rainbow H. Later, the anti-Ramsey numbers for several special graph classes in complete are determined. Moreover, researchers generalized the host graph K_n to other graphs, in particular, to complete bipartite graphs and regular bipartite graphs. Li and Xu (2009) [18] proved that: Let G be a k-regular bipartite graph with n vertices in each partite set, then $AR(G, mK_2) = k(m-2)+1$ for all $m \geq 2$, $k \geq 3$ and $n > 3(m-1)$. In this paper, we consider the anti-Ramsey number for matchings in 3-regular bipartite graphs. By using the known result that the vertex cover equals the size of maximum matching in bipartite graphs, we prove that $AR(G, mK_2) = 3(m-2)+1$ for $n > \frac{3}{2}(m-1)$ when G is a 3-regular bipartite graph with n vertices in each partite set.

1. Introduction

An edge-colored graph is called a *rainbow* graph if the colors on its edges are distinct. The rainbow generalizations of Ramsey theory are areas of current and very active research, one of which is the anti-Ramsey numbers. The anti-Ramsey number $AR(K_n, H)$ was introduced by Erdős Simonovits and Sós [6] in 1973, which is defined to be the maximum number of colors in an edge coloring of the complete graph K_n without any rainbow H. Note that researchers also used the notation *rainbow number* $rb(G, H)$ which is defined to be the minimum number r of colors such that every r-edge-coloring of G contains a rainbow copy of H. Clearly, $rb(G, H) = AR(G, H) + 1$.

It has been shown that the anti-Ramsey number $AR(K_n, H)$ is closely related to Turán number of the family $\{H - e : e \in E(H)\}$ in K_n for the graph H such that $\min\{\chi(H - e) : e \in E(H)\} \geq 3$. The anti-Ramsey numbers for some special graph classes in complete graph have been determined, including cycles [1,6,12,20], paths [22], complete graph [19,21], small bipartite graphs [2], stars [10], subdivided graphs [11], trees with k edges [13], graphs with independent cycles [15] and matchings [5,8,21].

Moreover, researchers generalized the host graph K_n to other graphs, in particular, to complete bipartite graphs and regular bipartite graphs. The bipartite anti-Ramsey number was studied for even cycles [3], stars [10], matchings [17], and trees with k edges [14]. There are few results for the case when the host graph G is a general graph. Interestingly, Li and Xu [18] considered the anti-Ramsey numbers of matchings in regular bipartite graphs. Recently, Jendrol, Schiermeyer and Tu

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[9] obtained bounds for the anti-Ramsey number for matching in plane triangulations. Xu et al. [23] studied the problem in general graphs. More results on anti-Ramsey numbers can be found in comprehensive surveys [7,16].

In this paper, we consider the anti-Ramsey number for matchings in 3-regular bipartite graphs. By using the known result that the vertex cover equals the size of maximum matching in bipartite graphs, we prove that $AR(G, mK_2) = 3(m-2)+1$ for $n > \frac{3}{2}(m-1)$ when G is a 3-regular bipartite graph with n vertices in each partite set. This improves the bound of Li and Xu [18].

2. Preliminaries

Let G be a graph and c be a coloring of $E(G)$. For an edge $e \in E(G)$, denote by $c(e)$ the color assigned to the edge e. Let $S \subseteq V(G)$, denote by $e_G(S)$ the number of edges incident to S in G. Let S, $T \subseteq V(G)$ and $S \cap T = \emptyset$, denote by $[S, T]_G$ the set of edges of G which has one end vertex in S and T respectively.

A subset S of $V(G)$ such that every edge of G has at least one end in S is called a *vertex cover* of G. The number of vertices in a minimum vertex cover of G is denoted by $\beta(G)$. The number of edges in a maximum matching of G is denoted by $\alpha'(G)$

Lemma 2.1 [4]**.** *For every bipartite graph G, $\alpha'(G) = \beta(G)$.*

Next we give the Turán-type result for matchings in 3-regular bipartite graphs. The proof can be found in [18] and here we omit the proof details.

Lemma 2.2 [18]**.** *Let G be a bipartite graph with $\Delta(G) \leq 3$. If $|E(G)| \geq 3m-2$, then G contains a matching of size m.*

Li and Xu [18] proved the following result.

Theorem 2.3 [18]**.** *Let G be a k-regular bipartite graph with n vertices in each partite set. For all $m \geq 2$, $k \geq 3$ and $n > 3(m-1)$, $AR(G, mK_2) = k(m-2)+1$.*

In this paper, by a kind of more precisely computation, we improve the bound for 3-regular bipartite graphs and obtain the following result.

Theorem 2.4. *Let $m \geq 2$ and G be a 3-regular bipartite graph with n vertices in each partite set. If $n > \frac{3}{2}(m-1)$, then $AR(G, mK_2) = 3m-5$.*

Sketch of Proof:

Denote by (X, Y) the bipartition of G with $|X| = |Y| = n$. First, we present a $(3m-5)$-edge-coloring of G which does not contain any rainbow matching of size m as follows. Take a $(m-2)$-subset S of X. Color all the edges incident to the vertices of S by distinct colors and color the rest edges by a new color. Clearly, there does not exist any rainbow matching of size m. So $AR(G, mK_2) \geq 3m-5$.

Now we prove the inequality $AR(G, mK_2) \leq 3m-5$. We only need to show that each $(3m-4)$-edge-coloring of G contains a rainbow matching of size m. We prove this by the contradiction hypothesis. Let c be a $(3m-4)$-edge-coloring of G which has no rainbow matching of size m. Let H be a rainbow spanning subgraph of G with $|E(H)| = 3m-4$. By the contradiction hypothesis, there is no mK_2 in H. By Lemmas 2.2 and 2.1, H has a minimum vertex cover V of size $m-1$. Let $S = V \cap X$ and $T = V \cap Y$. Let $|S| = s$ and $|T| = t$, then $s+t = m-1$. Take the graph H that maximizes the value of $|s-t|$. Without loss of generality, assume that $s \geq t$.

Since G is a 3-regular bipartite graph, one of the followings must hold.

Case 1. For any $x \in V$, $d_H(x) = 3$, and $|[S, T]_G| = 1$.
Case 2. There exists a vertex $v \in T$ such that $d_H(v) = 2$ and $d_H(x) = 3$ for any $x \in V \setminus \{v\}$, and $[S, T]_H = \emptyset$.
Case 3. There exists a vertex $u \in S$ such that $d_H(u) = 2$ and $d_H(x) = 3$ for any $x \in V \setminus \{u\}$, and $[S, T]_H = \emptyset$.

3. Proof of Case 1

Let $u \in S$ and $v \in T$ with $uv \in E(H)$. Then uv is a cut edge of H. Let H_1 and H_2 denote the unions of components of $H - uv$ which contain vertices of S and T, respectively.

Claim 1. Let $xy \in E(G) \setminus E(H)$ with $x \in N_{H_2}(T)$ and $c(xy) = c(e)$ for the edge $e \in E(H)$. Then $H_2 - x - e$ contains a matching of size t.

Let $H_2' = H_2 - x - e$, then $E(H_2') \geq 3t-4$. Suppose that H_2' does not have any rainbow matching of size t. By Lemmas 2.2 and 2.1, H_2' has a vertex cover V' of size $t-1$. Let $S_1 = V' \cap X$ and $T_1 = V' \cap Y$. Let $H' = H + xy - e$ and $S' = S \cup S_1 \cup \{x\}$. Then $S' \cup T_1$ is also a vertex cover of H' of size $m-1$. However $|S'| - |T_1| > |S| - |T|$, a contradiction.

Claim 2. For any $y \in Y$, $d_H(y) \geq 1$.

Suppose that there exists a vertex $y \in Y$ with $d_H(y) = 0$. Then there exists a vertex $x \in X \setminus S$ such that $xy \in E(G) \setminus E(H)$. Let $e \in E(H)$ such that $c(e) = c(xy)$.