Applied Mathematics and Computation 292 (2017) 120–127

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the automorphisms of $2-(v,k,1)$ designs

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ARTICLE INFO

MSC:
05B05
20B25

Keywords:
Block-transitive
$2-(v,k,1)$ design
Automorphism
Socle

ABSTRACT

Thirty years ago, a six-person team classified the pairs $(\mathcal{D},G)$ where $\mathcal{D}$ is a $2-(v,k,1)$ design and G is a flag-transitive automorphism group of $\mathcal{D}$, with the exception of those in which G is a one-dimensional affine group. Since then the effort has been to classify those $2-(v,k,1)$ designs which are block-transitive but not flag-transitive. This paper contributes to the program for determining the pairs $(\mathcal{D},G)$ in which $\mathcal{D}$ has a block-transitive group G of automorphisms. It is clear that if one wishes to study the structure of a finite group acting on a $2-(v,k,1)$ design then describing the socle is an important first step. Here we prove that if G is a block-transitive group of automorphisms of $\mathcal{D}$ which has $T={}^2F_4(q)$ as its socle then T is also transitive on the blocks of $\mathcal{D}$.

1. Introduction

This paper is part of a project to classify groups and $2-(v,k,1)$ designs where the group acts transitively on the blocks of the design. A $2-(v,k,1)$ design $\mathcal{D}=(\mathcal{P},\mathcal{B})$ is a pair consisting of a finite set $\mathcal{P}$ of points and a collection $\mathcal{B}$ of $k-$subsets of $\mathcal{P}$, called blocks, such that any 2-subsets of $\mathcal{P}$ is contained in exactly one block. Traditionally one defined $v=:|\mathcal{P}|$ and $b=:|\mathcal{B}|$. We will always assume that $2<k<v$.

Thirty years ago, a six-person team [4] classified the pairs $(\mathcal{D},G)$ where $\mathcal{D}$ is a $2-(v,k,1)$ design and G is a flag-transitive automorphism group of $\mathcal{D}$, with the exception of those in which G is a one-dimensional affine group. Since then the effort has been to classify those $2-(v,k,1)$ designs which are block-transitive but not flag-transitive. This paper contributes to the program for determining the pairs $(\mathcal{D},G)$ in which $\mathcal{D}$ has a block-transitive subgroup, G, of automorphisms. From the assumption that G is transitive on the set $\mathcal{B}$ of blocks, it follows that G is also transitive on the point set $\mathcal{P}$. This is a consequence of the theorem of Block in [2].

It is clear that if one wishes to study the structure of a finite group acting on a $2-(v,k,1)$ design then describing the socle is an important first step. In 1996 Camina showed in [5] that if G is a block-transitive, point-primitive automorphism group of a $2-(v,k,1)$ design $\mathcal{D}$, then the socle of G is either elementary abelian or simple. Camina and Praeger gave a generalization of this result that if G acts as a block-transitive, point-quasiprimitive automorphism group of a $2-(v,k,1)$ design $\mathcal{D}$, then G is affine or almost simple in [7]. Thus the study of block-transitive $2-(v,k,1)$ designs can be reduced to three cases, distinguishable by properties of the action on the point-set: that in which G is of affine type in the sense that it has an elementary abelian transitive normal subgroup; that in which G has an intransitive minimal normal subgroup; and that in which G is almost simple, in the sense that there is a simple transitive normal subgroup T in G whose centraliser is

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http://dx.doi.org/10.1016/j.amc.2016.06.027
0096-3003/

trivial, so that $T \trianglelefteq G \leq Aut(T)$. Camina and Spiezia have proved that T is not a sporadic simple group in [9]. Camina, Neumann, and Praeger showed that T cannot be an alternating group unless $G = A_7$ and A_8 in 2003 (see [6]). Liu et al. have studied the special case where $G = T = Soc(G)$ is a finite group of Lie type of Lie rank 1 acting block-transitively on a design in [18–21]. Recently, there have been a number contributions to this classification (see [14–16], [22–24]). Here we focus on classifying $2-(v, k, 1)$ designs admitting a block-transitive automorphism group of almost simple type with socle ${}^2F_4(q)$ and prove the following theorem:

Main Theorem Let G be an almost simple group acting block-transitively on a $2-(v, k, 1)$ design $\mathcal{D}$. If $T = Soc(G)$ is isomorphic to ${}^2F_4(q)$, then T is block-transitive.

We introduce some notation below. Let X and Y be arbitrary finite groups. The expression $X \cdot Y$ denotes an extension of X by Y and X: Y denotes the split extension. If Y is a subgroup of X, then the symbol $|X: Y|$ denotes the index of Y in X. The symbol $[m]$ denotes an arbitrary group of order m while Z_m or simply m denotes a cyclic group of that order. Other notation for group structure is standard. In addition, we use the symbol $p^i || n$ to denote $p^i | n$ but $p^{i+1} \nmid n$ and the symbol $\pi(n)$ to denote the set of prime divisors of an integer n. Let G be a finite group and $x \in G$. The symbol $Fix_{\Omega}(\langle x \rangle)$ denote the fixed points set of $\langle x \rangle$ acting on a set Ω. Let G be a group acting on a $2-(v, k, 1)$ design $\mathcal{D}$. The symbols G_α and G_B denote the stabilizer of point α of $\mathcal{P}$ and the stabilizer of block B of $\mathcal{B}$ in the action of G, respectively.

The paper is organized as follows. Section 2 describes several preliminary results concerning the group ${}^2F_4(q)$ and $2-(v, k, 1)$ designs. Section 3 gives the proof of the theorem.

2. Preliminary results

The Ree groups ${}^2F_4(q)$ are the fixed points of a certain automorphism of the Chevalley groups of type F_4 over a finite field $F = GF(q)$, where $q = 2^a$ with $a = 2n + 1, n \geq 0$. Ree [27] showed that the groups ${}^2F_4(q)$ are simple if $q > 2$, while Tits [28] showed that ${}^2F_4(2)$ is not simple but possesses a simple subgroup of index 2. Let $T = {}^2F_4(q)$. Then the order of T is $q^{12}(q-1)(q^3+1)(q^4-1)(q^6+1)$.

Let both G and A be finite groups, and Ω a finite set. A triple (A, G, Ω) is called *exceptional* if it satisfies the following conditions:

(1) G is a normal subgroup of A;
(2) both A and G are transitive permutation groups on Ω;
(3) the diagonal of $\Omega \times \Omega$ is the only common orbit of A and G on $\Omega \times \Omega$.

This definition is equivalent to the following: Let $\alpha \in \Omega$, then every A_α-orbit except $\{\alpha\}$ breaks up into strictly smaller G_α-orbits.

We call the triple (A, G, Ω) *arithmetically exceptional*, if there is a subgroup B of A which contains G, such that (B, G, Ω) is exceptional, and B/G is cyclic. When A is a primitive permutation group of almost simple type, Guralnick, Muller and Saxl have obtained their classification (see [12]). In particular, when $Soc(A) = {}^2F_4(q)$, there is the following lemma:

Lemma 2.1 (Theorem 1.5 (g) of [12])**.** *Let A be a primitive permutation group of almost simple type, and $L \trianglelefteq A \leq Aut(L)$ with $L = {}^2F_4(q)$. Suppose that there are subgroups B and G of A with $G \trianglelefteq A$ and B/G cyclic, such that (B, G) is exceptional. Let M be a point stabilizer in A. Then $M \cap L$ is a subfield group, the centralizer in L of a field automorphism of odd prime order r.*

Lemma 2.2 [26]**.** *Every maximal subgroup of $T = {}^2F_4(q)$, $q = 2^{2n+1}$, $n \geq 1$, is conjugate to one of the following Table 1:*

Table 1
Maximal subgroups of $T = {}^2F_4(q)$.

Structure	Order	Remarks
$P_1 = [q^{11}] : (PSL(2, q) \times (q-1))$	$q^{12}(q+1)(q-1)^2$	*parabolic*
$P_2 = [q^{10}] : ({}^2B_2(q) \times (q-1))$	$q^{12}(q-1)^2(q^2+1)$	*parabolic*
$SU(3, q)$: 2	$2q^3(q-1)(q+1)^2(q^2-q+1)$	
$(Z_{q+1} \times Z_{q+1}) : GL(2, 3)$	$48(q+1)^2$	
$(Z_{q+\epsilon\sqrt{2q}+1} \times Z_{q+\epsilon\sqrt{2q}+1}) : [96]$	$96(q+\epsilon\sqrt{2q}+1)^2$	if $\epsilon = -$, $q > 8$
$Z_{q^2+\epsilon\sqrt{2}q^{\frac{3}{2}}+q+\epsilon\sqrt{2q}+1} : 12$	$12(q^2+\epsilon\sqrt{2}q^{\frac{3}{2}}+q+\epsilon\sqrt{2q}+1)$	$\epsilon = \pm$
$PGU(3, q)$: 2	$2q^3(q-1)(q+1)^2$	
${}^2B_2(q) \wr 2$	$2q^2(q^2+1)(q-1)$	
$B_2(q)$: 2	$2q^4(q^2-1)(q^4-1)$	
${}^2F_4(q_0)$	$q_0^6(q_0-1)(q_0^3+1)(q_0^4-1)(q_0^6+1)$	$q = q_0^\delta$ and δ is a prime

Conversely, if H is conjugate to one of these groups, then $N_G(H)$ is maximal in G.

Lemma 2.3 ([1], Proposition 2.34)**.** *The Ree group ${}^2F_4(q)$ has only two conjugacy classes of involutions, and the orders of the centralizers of involutions $j_1 = 2A$, $j_2 = 2B$ are $q^{12}(q-1)(q^2+1)$ and $q^{10}(q^2-1)$, respectively.*

Lemma 2.4. *Let G be a finite group and H a subgroup of G, $x \in H$. Then $|C_G(x)| \leq |C_H(x)||G: H|$.*