Applied Mathematics and Computation 292 (2017) 156–164

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Group analysis of a hyperbolic Lane–Emden system

Tshepo E. Mogorosi [a], Igor L. Freire [b], Ben Muatjetjeja [a], Chaudry Masood Khalique [a,*]

[a] International Institute for Symmetry Analysis and Mathematical Modelling, Department of Mathematical Sciences, North-West University, Mafikeng Campus, Private Bag X 2046, Mmabatho 2735, South Africa

[b] Centro de Matemática, Computação e Cognição Universidade Federal do ABC-UFABC, Rua Santa Adélia, 166, Bairro Bangu, 09.210-170, Santo André, SP, Brasil

ARTICLE INFO

Keywords:
Lie symmetries
Noether symmetries
Conservation laws
Hyperbolic Lane–Emdem system
Exact solutions

ABSTRACT

In this paper we carry out a complete Noether and Lie group classification of the radial form of a coupled system of hyperbolic equations. From the Noether symmetries we establish the corresponding conserved vectors. We also determine constraints that the nonlinearities should satisfy in order for the scaling symmetries to be Noetherian. This led us to a critical hyperbola for the systems under consideration. An explicit solution is also obtained for a particular choice of the parameters.

1. Introduction

The study of the coupled elliptic equations

$$\begin{cases} \Delta u + v^q = 0, \\ \Delta v + u^p = 0, \end{cases} \tag{1.1}$$

called Lane–Emden systems, is an active branch in nonlinear analysis. In (1.1), $u = u(x)$, $v = v(x)$ and $x \in \mathbb{R}^n$. Such a system, particularly when $n \geq 3$ and $p, q > 0$, has been widely investigated from different point of views. See for example [14,16,18,25].

System (1.1) can be considered as a natural generalisation of the celebrated Lane–Emden equation

$$\Delta u + u^p = 0, \tag{1.2}$$

where $u = u(x)$, $x \in \mathbb{R}^n$ [18]. Eq. (1.2) has a “natural hyperbolic partner”, given by the nonlinear wave equation

$$u_{tt} - \Delta u - u^p = 0, \tag{1.3}$$

where $(t, x) \in \mathbb{R}^{1+n}$ and $u = u(t, x)$. In this case, t can be interpreted as a time variable, while x corresponds to the spatial ones. It is then natural to consider the hyperbolic generalisation of (1.1), which we shall refer to as hyperbolic Lane–Emden system, given by

$$\begin{cases} \tilde{u}_{tt} - \Delta \tilde{u} - \tilde{v}^q = 0, \\ \tilde{v}_{tt} - \Delta \tilde{v} - \tilde{u}^p = 0. \end{cases} \tag{1.4}$$

* Corresponding author.
E-mail addresses: tshepomogorosi@rocketmail.com (T.E. Mogorosi), igor.freire@ufabc.edu.br, igor.leite.freire@qmail.com (I.L. Freire), Ben.Muatjetjeja@nwu.ac.za (B. Muatjetjeja), masood.khalique@nwu.ac.za, masoodkhalique@gmail.com (C.M. Khalique).

http://dx.doi.org/10.1016/j.amc.2016.07.033
0096-3003/

If we define $r := \|x\|$, $\tilde{u}(t,x) = u(t,r)$ and $\tilde{v}(t,x) = v(t,r)$, system (1.4) can therefore be rewritten in its *radial* form as

$$\begin{cases} u_{tt} - u_{rr} - \dfrac{n-1}{r}u_r - v^q = 0, \\ v_{tt} - v_{rr} - \dfrac{n-1}{r}v_r - u^p = 0. \end{cases} \tag{1.5}$$

A simple generalisation of (1.5) can easily be obtained if we replace the integer $n-1$, related to the dimension on the space of spatial coordinates, by an arbitrary real parameter ν.

Thus, in this paper we consider the following hyperbolic version of the Lane–Emden system:

$$\begin{cases} u_{tt} - u_{rr} - \dfrac{\nu}{r}u_r - v^q = 0, \\ v_{tt} - v_{rr} - \dfrac{\nu}{r}v_r - u^p = 0 \end{cases} \tag{1.6}$$

from the point of view of Lie group analysis.

As far as we know, it was the PhD thesis of Gilli Martins [15], and the works arisen from there (see [4] and references therein), that started the investigation of symmetry properties of the Lane–Emden systems in the sense of Lie symmetry theory [1–3,17,24]. Since then several works have been done in this direction. See for example [8–10,12,19–21].

If at least one of the powers in (1.4) is 0 we obtain a coupled system such that one of the equations satisfies $w_{tt} - \Delta w - 1 = 0$, which does not have any dependence with respect to the other variable and leads us to an uninteresting case. On the other hand, if at least one of them is 1, say q, we can consequently obtain the biwave equation $\Box^2 w - w^p = 0$, where $\Box := \partial_{tt} - \Delta$, a case already investigated, from the point of view of Lie symmetries, in [13]. For this reason, in this paper we assume that $p, q \neq 0, 1$. This condition is the only one to be assumed regarding the nonlinearities.

With regard to the parameter ν, we only assume that it is different from 0. Actually, the case $\nu = 0$ can either be obtained from [8], under the complex transformation $(x, y, u, v) \mapsto (t, ir, u, v)$ into the original variables of the mentioned reference, or from [21], by making use of projections on the $(t, x)-$space once it is assumed that in [21] the functions u and v depend only on (t, x) instead of (t, x, y). In this case, the symmetries for system (1.6), with $\nu = 0$, for any p and q, are given by

$$T = \frac{\partial}{\partial t}, \quad R = \frac{\partial}{\partial r}, \quad H = r\frac{\partial}{\partial t} + t\frac{\partial}{\partial r}. \tag{1.7}$$

For other generators, depending on the powers p and q, see [21].

The outline of the paper is as follows. In Section 2 we revisit the main points concerning Noether symmetries and how to construct conserved vector once a Noether symmetry is known. Next, in Section 3, the Noether symmetries are found and their corresponding conserved vectors are established. The complete group classification of (1.6) is obtained in Section 4. Discussions about the results obtained in the Sections 3 and 4, as well as their connections with the results obtained in [8–10,21], are presented in Section 5.

2. Preliminaries on Noether symmetry

In this section we give some salient features of Noether symmetries concerning the system of two second-order partial differential equations (PDEs), which we use in Section 3. For more details see for example [19,22].

We now consider the vector field

$$X = \tau(t,r,u,v)\frac{\partial}{\partial t} + \xi(t,r,u,v)\frac{\partial}{\partial r} + \eta^1(t,r,u,v)\frac{\partial}{\partial u} + \eta^2(t,r,u,v)\frac{\partial}{\partial v}. \tag{2.8}$$

The first-order prolongation of X is given by

$$X^{[1]} = \tau\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial x} + \eta^1\frac{\partial}{\partial u} + \eta^2\frac{\partial}{\partial v} + \zeta_t^1\frac{\partial}{\partial u_t} + \zeta_t^2\frac{\partial}{\partial v_t} + \zeta_r^1\frac{\partial}{\partial u_r} + \zeta_r^2\frac{\partial}{\partial v_r}, \tag{2.9}$$

where

$$\begin{aligned} \zeta_t^1 &= D_t(\eta^1) - u_t D_t(\tau) - u_r D_t(\xi), \quad \zeta_r^1 = D_r(\eta^1) - u_t D_r(\tau) - u_r D_r(\xi), \\ \zeta_t^2 &= D_t(\eta^2) - v_t D_t(\tau) - v_r D_t(\xi), \quad \zeta_r^2 = D_r(\eta^2) - v_t D_r(\tau) - v_r D_r(\xi) \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} D_t &= \frac{\partial}{\partial t} + u_t\frac{\partial}{\partial u} + v_t\frac{\partial}{\partial v} + u_{tt}\frac{\partial}{\partial u_r} + v_{tt}\frac{\partial}{\partial v_r} + u_{tr}\frac{\partial}{\partial u_r} + v_{tr}\frac{\partial}{\partial v_r} + \cdots, \\ D_r &= \frac{\partial}{\partial r} + u_r\frac{\partial}{\partial u} + v_r\frac{\partial}{\partial v} + u_{rr}\frac{\partial}{\partial u_r} + v_{rr}\frac{\partial}{\partial v_r} + u_{tr}\frac{\partial}{\partial u_t} + v_{tr}\frac{\partial}{\partial v_t} + \cdots. \end{aligned} \tag{2.11}$$

Recall that the Euler–Lagrange operators are defined by