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Asymptotic formulae for solutions of half-linear differential equations

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ABSTRACT

We establish asymptotic formulae for regularly varying solutions of the half-linear differential equation

$$(r(t)|y'|^{\alpha-1}\operatorname{sgn} y')' = p(t)|y|^{\alpha-1}\operatorname{sgn} y,$$

where r, p are positive continuous functions on $[a, \infty)$ and $\alpha \in (1, \infty)$. The results can be understood in several ways: Some open problems posed in the literature are solved. Results for linear differential equations are generalized; some of the observations are new even in the linear case. A refinement on information about behavior of solutions in standard asymptotic classes is provided. A precise description of regularly varying solutions which are known to exist is given. Regular variation of all positive solutions is proved.

1. Introduction

We consider the half-linear equation

$$(r(t)\Phi(y'))' = p(t)\Phi(y), \tag{1}$$

where r, p are positive continuous functions on $[a, \infty)$ and $\Phi(u) = |u|^{\alpha-1}\operatorname{sgn} u$ with $\alpha > 1$. We are interested in asymptotic behavior of solutions of (1); we obtain asymptotic formulae for (normalized) regularly varying solutions.

Our results can be understood in several ways. We solve open problems posed in the literature ([16,17]). We generalize results for linear differential equations ([6]); some observations are new even in the linear case. We provide a refinement on information about behavior of solutions in standard asymptotic classes ([2,3]). We give a precise description of regularly varying solutions which are known to exist ([3,10,11,16]).

That the theory of regularly varying functions can be very useful in the study of asymptotic properties of differential equations has been shown in many works, see the monograph [12] and the survey text [15]. Half-linear differential equations were studied in this framework e.g. in [3,9–11,13,14,16,17].

The paper is organized as follows. In the next section we recall basic information on nonoscillatory solutions of (1). Section 3 is devoted to asymptotic formulae in the case $\lim_{t\to\infty} t^{\alpha} p(t)/r(t) = 0$. In particular we recall existing results there, which serve to prove complementary result and generalizations. The case $\lim_{t\to\infty} t^{\alpha} p(t)/r(t) = C > 0$ is treated in Section 4. A modified Riccati technique plays an important role in the proof. We discuss also necessary conditions and relations to

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standard asymptotic classes. A similar idea is used in Section 5 to establish a new proof for a variant of the result from Section 3. Section 6 is devoted to a generalization of the previous results, based on suitable transformations of independent variable. The last section is an Appendix with basic information on the theory of regular variation, which plays a significant role in this paper.

2. Nonoscillatory solutions

It is known (see [3, Chapter 4]) that (1) with positive r, p is nonoscillatory, i.e. all its solutions are eventually of constant sign. Without loss of generality, we work just with positive solutions, i.e. with the class

$$\mathcal{S}=\{y : y(t) \text{ is a positive solution of } (1) \text{ for large } t\}.$$

Because of the sign conditions on the coefficients, all positive solutions of (1) are eventually monotone, therefore they belong to one of the following disjoint classes:

$$\mathcal{IS}=\{y\in\mathcal{S} : y'(t)>0 \text{ for large } t\},\quad \mathcal{DS}=\{y\in\mathcal{S} : y'(t)<0 \text{ for large } t\}.$$

It can be shown that both these classes are nonempty (see ([3, Lemma 4.1.2]). The classes $\mathcal{IS},\mathcal{DS}$ can be divided into four mutually disjoint subclasses:

$$\mathcal{IS}_\infty=\{y\in\mathcal{IS} : \lim_{t\to\infty}y(t)=\infty\},\quad \mathcal{IS}_B=\{y\in\mathcal{IS} : \lim_{t\to\infty}y(t)=b\in\mathbb{R}\},$$
$$\mathcal{DS}_B=\{y\in\mathcal{DS} : \lim_{t\to\infty}y(t)=b>0\},\quad \mathcal{DS}_0=\{y\in\mathcal{DS} : \lim_{t\to\infty}y(t)=0\}.$$

Define the so-called quasiderivative of $y\in\mathcal{S}$ by $y^{[1]}=r\Phi(y')$. We introduce the following convention

$$\mathcal{IS}_{u,v}=\{y\in\mathcal{IS} : \lim_{t\to\infty}y(t)=u,\quad \lim_{t\to\infty}y^{[1]}(t)=v\}$$
$$\mathcal{DS}_{u,v}=\{y\in\mathcal{DS} : \lim_{t\to\infty}y(t)=u,\quad \lim_{t\to\infty}y^{[1]}(t)=v\}.$$

For subscripts of $\mathcal{IS}$ and $\mathcal{DS}$, by $u=B$ resp. $v=B$ we mean that the value of u resp. v is a real nonzero number. Using this convention we further distinguish the following types of solutions which form subclasses in $\mathcal{DS}_0,\mathcal{DS}_B,\mathcal{IS}_B$, and $\mathcal{IS}_\infty$:

$$\mathcal{DS}_{0,0},\mathcal{DS}_{0,B},\mathcal{DS}_{B,0},\mathcal{DS}_{B,B},\mathcal{IS}_{B,B},\mathcal{IS}_{B,\infty},\mathcal{IS}_{\infty,B},\mathcal{IS}_{\infty,\infty}. \tag{2}$$

More information about (non)existence of solutions in these subclasses can be found in [2] and [3, Chapter 4]. In some places we need to emphasize that the classes of eventually positive increasing resp. decreasing solutions resp. their subclasses are associated with a particular equation, say (∗). Then we write $\mathcal{IS}^{(*)}$, $\mathcal{DS}^{(*)}$, $\mathcal{IS}^{(*)}_\infty$, etc.

No matter whether p is positive, if (1) is nonoscillatory, then there exists a nontrivial solution y of (1) such that for every nontrivial solution u of (1) with $u\neq\lambda y$, $\lambda\in\mathbb{R}$, we have $y'(t)/y(t)<u'(t)/u(t)$ for large t, see [3, Section 4.2]. Such a solution is said to be a principal solution. Solutions of (1) which are not principal, are called nonprincipal solutions. Principal solutions are unique up to a constant multiple. Denote $\mathfrak{P}=\{y\in\mathcal{S} : y \text{ is principal}\}$.

Let $y\in\mathcal{S}$. Denoted $w=r\Phi(y'/y)$, it satisfies the generalized Riccati equation

$$w'-p(t)+(\alpha-1)r^{1-\beta}(t)|w|^\beta=0, \tag{3}$$

where β denotes the conjugate number of α, i.e., $1/\alpha+1/\beta=1$, see [3, Chapter 1]. Another substitution (introduced in [4]) $v=h^\alpha w-rh\Phi(h')$, $h\in C^1$, $h(t)\neq 0$ leads to a modified Riccati equation (see (15) below), and is very useful in the proof of Theorem 2. This substitution supplies—to some extent—the ("linear") transformation of dependent variable $y=hu$ in Eq. (1), which does not work in the half-linear case because of lack of additivity. Recall that another serious limitation in the theory of half-linear differential equations is the absence of a reduction of order formula; the reason is that there is no reasonable Wronskian identity for half-linear equations.

By Φ^{-1} we mean the inverse of Φ, i.e., $\Phi^{-1}(u)=|u|^{\beta-1}\operatorname{sgn}u$. If $\alpha=2$, then $\Phi=\Phi^{-1}=\mathrm{id}$ and (1) reduces to the linear equation $(r(t)y')'=p(t)y$.

3. The case $t^\alpha p(t)/r(t)\to 0$

For the notation concerning regular variation (like $\mathcal{RV}(\vartheta)$, $\mathcal{NRV}(\vartheta)$, $\mathcal{SV}$, $\mathcal{NSV}$, L_f, $\mathcal{RV}_\omega(\vartheta)$, etc.) which is used throughout the paper, see Appendix. As usual, the relation $f(t)\sim g(t)$ (as $t\to\infty$) means $\lim_{t\to\infty}f(t)/g(t)=1$ and $f(t)=o(g(t))$ (as $t\to\infty$) means $\lim_{t\to\infty}f(t)/g(t)=0$. To simplify writing of many asymptotic formulae, we denote

$$\mathfrak{E}(\sigma,\tau,C,f)=\exp\left\{\int_\sigma^\tau(1+o(1))Cf(s)\,\mathrm{d}s\right\},$$

where $o(1)$ is meant either as $\tau\to\infty$ when $\tau<\infty$ or as $\sigma\to\infty$ when $\tau=\infty$.

The following conditions play an important role:

$$p\in\mathcal{RV}(\delta),\quad r\in\mathcal{RV}(\delta+\alpha) \tag{4}$$