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Distributed constraint optimization on networked multi-agent systems

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ABSTRACT

This paper deals with a distributed constraint optimization problem on networked multi-agent systems. First, we propose a distributed algorithm based on the Lagrangian method, where a new update law of the Lagrangian multiplier is designed. This update law enables each agent to estimate the value of the Lagrangian multiplier in a distributed manner. Next, we derive a necessary and sufficient condition that the optimization problem is solvable in a distributed manner over a graph. Finally, we apply the proposed method to power grid control via distributed pricing to maintain the supply-demand balance.

1. Introduction

Recently, distributed optimization has been vigorously investigated to solve problems in large-scale systems. The most successful example is the Internet protocols [1–4], where congestion control is implemented. Distributed control techniques are inevitable to reduce computation burdens in such an enormously large-scale system. In recent years, these techniques are becoming more and more important because the sizes of many public infrastructures are growing (e.g., power grids [5–8] and traffic systems [9–11]) and the management of these systems is getting harder.

This paper deals with a distributed optimization problem in the terminology of networked multi-agent systems. Let $x_i \in \mathbb{R}^{d_i}$ be the variable updated by agent $i \in \{1, 2, \dots, n\}$. For $x = [x_1^\top x_2^\top \cdots x_n^\top]^\top$, let $F(x) \in \mathbb{R}$ and $G_j(x) \in \mathbb{R}$, $j = 1, 2, \dots, m$ be the objective and constraint functions. Then, the constraint optimization problem is formulated as follows.

$$\begin{cases} \text{minimize} & F(x) \\ \text{subject to} & G_j(x) \le 0,\ j = 1, 2, \dots, m \end{cases} \tag{1}$$

Note that equality constraints can be easily handled by the choice of $G_j(x)$. Now, these agents can communicate with each other over a network $\mathcal{G}$ (which is described by a graph).

Then, each agent can update the value of x_i by using only local information over $\mathcal{G}$. Let $x_i[k] \in \mathbb{R}^{d_i}$ be the updated value of x_i at time k, and $\lambda_j \in \mathbb{R}$ be the Lagrangian multiplier (a kind of penalty) according to the constraint $G_j(x)$. Then, the distributed update law based on the Lagrangian method is of the form

$$x_i[k+1] = f_i(x_i[k], x_{i_1}[k], x_{i_2}[k], \dots, x_{i_{n_i}}[k], \lambda_1, \lambda_2, \dots, \lambda_m), \tag{2}$$

where $i_1, i_2, \dots, i_{n_i}$ are the neighbors of agent i over the network $\mathcal{G}$. (2) is distributed in the sense that the agent requires only its own, its neighbors' information and the Lagrangian multipliers.

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Note that the Lagrangian multiplier λ_j is common among the agents, hence it is global information. As long as the update law (2) requires the information on λ_j, a supervisor is inevitable to compute and broadcast the value of λ_j. In a large-scale system, a huge amount of information will be concentrated on the supervisor. Thus, the update law (2) is not completely distributed nor scalable. In order to make the update law (2) distributed, the value of λ_j should be individually computed by each agent. However, generally, global information, the value of $G_j(x)$, is required to update λ_j. Is it possible to obtain the solution of (1) by distributed computations of not only x_i but also λ_j?

In this paper, we propose a new distributed algorithm to solve the optimization problem (1) over the network $\mathcal{G}$. Especially, we propose a distributed method to estimate the Lagrangian multiplier λ_j. By using this method, agents can individually obtain the estimated values of λ_j through information exchange between the agents. Then, whether the solution of (1) is obtained or not depends on the structure of $\mathcal{G}$. So, we derive a necessary and sufficient condition of $\mathcal{G}$ with which the problem (1) is solvable in the distributed way. From this result, if an optimization problem is given, we can answer a strict class of networks with which the problem is distributedly solvable. Conversely, if a network is given, we can know what kind of optimization problem is solvable over this network.

As an application of the proposed method, we consider power grid control via pricing to maintain supply-demand balance. The Lagrangian multiplier λ_j is corresponding to the electricity price. Our proposed method gives a basic rule to determine the price by negotiations between consumers, suppliers and distributors over a communication network.

1.1. Contributions comparing to related researches

Distributed optimization methods are summarized in [12]. For example, dual decomposition was developed in 1960s and has been modified by several researchers [13–16]. The main idea of the dual decomposition is to deal with separable functions with respect to agents, namely,

$$\begin{cases} \text{minimize} & \sum_{i=1}^{n} F_i(x_i) \\ \text{subject to} & \sum_{i=1}^{n} G_{ij}(x_i) \le 0,\ j = 1, 2, \ldots, m \end{cases} \tag{3}$$

where $F_i(x_i)$ and $G_{ij}(x_i)$ depend on one agent's variable x_i. Then, the update law of each agent is given by

$$x_i[k+1] = x_i[k] - \alpha\left(\frac{\partial F_i}{\partial x_i}(x_i[k]) + \sum_{j=1}^{m} \lambda_j \frac{\partial G_{ij}}{\partial x_i}(x_i[k])\right), \tag{4}$$

where α is a step gain. The update law (4) is of the form (2) because other agents' information is not required. However, this update law requires the global information λ_j. Moreover, the objective and constraint functions are restricted to separable forms as (3). The other primal-dual methods including alternating direction method of multipliers (ADMM), alternating direction augmented Lagrangian (ADAL) and so on [17,18] have the same structure.

Recently, consensus-based distributed optimization has been investigated [19–23]. The consensus-based distributed optimization for the objective function $F(x) = \sum_{i=1}^{n} F_i(x)$ with agent-wise functions $F_i(x)$ is reduced to

$$\begin{cases} \text{minimize} & \sum_{i=1}^{n} F_i(x_i) \\ \text{subject to} & x_i = x_j,\ i, j = 1, 2, \ldots, n \end{cases} \tag{5}$$

by introducing the constraints $x_i = x_j$. Then, the update law is given by

$$x_i[k+1] = x_i[k] - \alpha\left(\frac{\partial F_i}{\partial x_i}(x_i[k]) - \sum_{j \in \mathcal{N}_i} (x_i[k] - x_j[k])\right), \tag{6}$$

where $\mathcal{N}_i$ is the set of neighbors of agent i over the network $\mathcal{G}$. The update law (6) is distributed and the solution is obtained only if $\mathcal{G}$ is connected. Note that the objective function in (5) is separable, and other types of objective functions cannot be treated in this form.

Since these existing researches restrict optimization problems to the forms (3) and (5), the problems can be solved by the distributed update laws (4) and (6) (although (4) is not completely distributed because of λ_j). The question here is that *if we do not restrict optimization problems to these forms, what kind of problems can be solved in a distributed manner over the network $\mathcal{G}$? Are there more problems solvable in a distributed manner?* To answer these questions, we give a complete characterization of the class of optimization problems solvable in a distributed manner over $\mathcal{G}$. The derived class indicates the limit of the graph $\mathcal{G}$'s ability to solve distributed optimization problems. As far as the authors' knowledge, this is the first paper which specifies the graph's ability in this way. As a result, this paper reveals that we can solve a wider class of optimization problems than (3) and (5) in a distributed manner. See Example 1 and Section 4 for such examples.

1.2. Notations

The following notations are used in this paper. Let $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{Z}_+$ be the sets of the real numbers, nonnegative real numbers and nonnegative integers. The set of n-dimensional (non-negative) real numbers is represented as $\mathbb{R}^n$ ($\mathbb{R}_+^n$). The cardinal