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On edge-rupture degree of graphs

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ABSTRACT

The edge-rupture degree of an incomplete connected graph G is defined as $r'(G) = max\{\omega(G-S) - |S| - m(G-S) : S \subseteq E(G), \omega(G-S) > 1\}$, where $\omega(G-S)$ and $m(G-S)$, respectively, denote the number of components and the order of a largest component in $G-S$. This is a reasonable parameter to measure the vulnerability of networks, as it takes into account both the amount of work done to damage the network and how badly the network is damaged. In this paper, firstly, the relationships between the edge-rupture degree and some other graph parameters, namely the edge-connectivity, edge-integrity, edge-toughness, edge-tenacity, diameter, the algebraic connectivity and the minimum degree are established. After that, the edge-rupture degree of the middle graphs of path and cycle are given. Then, we introduced the concept of r'-maximal graph and give some basic results of such graphs. Finally, we introduce the concept of edge-ruptured and strictly edge-ruptured graph, and we establish necessary and sufficient conditions for a graph to be edge-ruptured and strictly edge-ruptured, respectively.

1. Introduction

The vulnerability of a communication network, composed of processing nodes and communication links, is of prime importance to network designers. As the network begins losing links or nodes, eventually there is a loss in its effectiveness. Thus, communication networks must be constructed to be as stable as possible, not only with respect to the initial disruption, but also with respect to the possible reconstruction of the network.

The communication network can be represented as an undirected and unweighted graph, where a processor (station) is represented as a vertex and a communication link between processors (stations) as an edge between corresponding vertices. If we use a graph to model a network, there are many graph theoretical parameters used to describe the vulnerability of communication networks.

Most notably, the connectivity and edge-connectivity have been frequently used. The difficulty with these parameters is that they do not take into account what remains after the graph is disconnected. Consequently, a number of other parameters have been introduced that attempt to cope with this difficulty, including integrity and edge-integrity in [3–6], toughness and edge-toughness in [8,12,17,25], tenacity and edge-tenacity in [13,20,28,30], rupture degree and edge-rupture degree in [20–23], and scattering number in [31,32]. Unlike the connectivity measures, each of these parameters shows not only the difficulty to break down the network but also the damage that has been caused. Quite a lot of different approaches to capture the robustness(vulnerability) properties of a network have been undertaken. One such category is

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the distance-based descriptors which include Wiener index, Wiener polarity index, Randić index and generalizations of the Randić index [19,24,29].

Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. For $S \subseteq E(G)$, let $\omega(G-S)$ and $m(G-S)$, respectively, denote the number of components and the order of a largest component in $G-S$. A set $S \subseteq E(G)$ is an *edge-cut set* of G, if $G-S$ is disconnected. We also use p and q to present the number of vertices(order) and the number of edges(size), respectively, of a graph G. $\lambda(G)$ denotes the *edge connectivity* of graph G. $diam(G)$ denotes the *diameter* of G. And we let $\delta(G)$ denote the *minimum degree* of G. We use Bondy and Murty [7] for terminology and notations not defined here. We recall some parameters as follows:

The *edge-connectivity* of an incomplete connected graph G:

$$\lambda(G) = min\{|S| : S \subset E(G), \omega(G-S) > 1\}.$$

The *edge-toughness* [8] of an incomplete connected graph G:

$$\tau_1(G) = min\left\{\frac{|S|}{\omega(G-S)-1} : S \subset E(G), \omega(G-S) > 1\right\}.$$

An edge-cut set S of a graph G is called a τ_1*-set* of G if it satisfies that

$$\tau_1(G) = \frac{|S|}{\omega(G-S)-1}.$$

The *edge-integrity* [3] of a graph G:

$$I'(G) = min\{|S| + m(G-S) : S \subset E(G)\}.$$

The *edge-tenacity* [28] of a graph G:

$$T'(G) = min\left\{\frac{|S| + m(G-S)}{\omega(G-S)} : S \subseteq E(G), \omega(G-S) > 1\right\}.$$

Now we introduce the concept of the *edge-rupture degree* of a graph G.

Definition 1.1 [23]**.** The edge-rupture degree of an incomplete connected graph G is defined as

$$r'(G) = max\{\omega(G-S) - |S| - m(G-S) : S \subseteq E(G), \omega(G-S) > 1\}.$$

As we know that, two ways of measuring the vulnerability of a network is through the ease with which one can disrupt the network, and the cost of a disruption. One can say that the disruption is more successful if the disconnected network contains more components and much more successful, if in addition, the affected components are small. One can associate the cost with the number of the edges destroyed to get small components and the reward with the number of the components remaining after destruction. The edge-rupture degree measure is compromise between the cost and the reward by minimizing the cost: reward ratio. Thus, a network with a small edge-rupture degree performs better under external attack.

Given an edge-cut set $S \subseteq E(G)$, the *score* of S is defined as

$$sc(S) = \omega(G-S) - |S| - m(G-S).$$

So, the edge-rupture degree of an incomplete connected graph G is also defined as

$$r'(G) = max\{sc(S) : S \subseteq E(G), \omega(G-S) > 1\},$$

where the maximum is taken over all edge-cut sets S of G.

A graph G is called *edge-ruptured* if $r'(G) = sc(E(G))$. A graph G is called *strictly edge-ruptured* if E is the unique edge set whose score equals $r'(G)$. Edge-ruptured graphs are somewhat analogous to edge-tenacious graphs in [28] and honest graphs in [3]. They can be considered very stable, because to minimize the ratio of cost to reward, the attacker needs to be destroy all of the edges in the networks, thus, attack tends to be "expensive" and so the networks are relatively invulnerable.

The concept of edge-rupture degree was first introduced in [23], where the authors give some basic results of this parameter as follows.

Theorem 1.1 [23]**.** *If G is connected and $S \subseteq E$, then $sc(S) \leq 0$ with equality if and only if G is a tree and $S = E$.*

Corollary 1.1 [23]**.** *If G is connected, then $r'(G) \leq 0$ with equality if and only if G is a tree.*

Theorem 1.2 [23]**.** *If H is a spanning subgraph of G, then $r'(G) \leq r'(H)$.*

Theorem 1.3 [23]**.** *For the cycle C_p, $r'(C_p) = -1$.*

Theorem 1.4 [23]**.** *For the complete bipartite graph $K_{m,n}$,*

$$r'(K_{m,n}) = (m+n) - mn - 1.$$