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A Runge–Kutta discontinuous Galerkin scheme for hyperbolic conservation laws with discontinuous fluxes

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ABSTRACT

The paper proposes a scheme by combining the Runge–Kutta discontinuous Galerkin method with a δ-mapping algorithm for solving hyperbolic conservation laws with discontinuous fluxes. This hybrid scheme is particularly applied to nonlinear elasticity in heterogeneous media and multi-class traffic flow with inhomogeneous road conditions. Numerical examples indicate the scheme's efficiency in resolving complex waves of the two systems. Moreover, the discussion implies that the so-called δ-mapping algorithm can also be combined with any other classical methods for solving similar problems in general.

1. Introduction

The standard hyperbolic conservation laws can be generally written in the following form [11,33,36]:

$$u_t + f(u)_x = 0, \tag{1}$$

where $u = u(x, t)$ is an unknown variable or vector for solution, and $f(u)$ is the flux. However, considerable applications involve spatially varying fluxes, e.g., in flow through porous media, water wave equations, elastic waves in heterogeneous media, and traffic flow on an inhomogeneous road. See [2,6,17,18,21,23–25,38,39,41,43] and the references therein for discussions of the problem. In this case, the equation or system for conservation is written as

$$u_t + f(u, \theta(x))_x = 0, \tag{2}$$

where $\theta(x)$ is a known scalar or vector denoting some spatially varying parameters.

For standard conservation laws of Eq. (1), study of numerical schemes focuses on the capture of shocks. Although the first-order monotone scheme is able to resolve a shock, the profile can be over smoothed by numerical diffusions. The Godunov theorem suggests that a high-order accurate linear scheme can considerably reduce these diffusions. However, the dispersion that is due to the linearity yields spurious oscillations in the vicinity of a shock. Thus, nonlinearity was introduced in the high-order accurate scheme to suppress the oscillations, with the proposition of the total variation diminishing (TVD) scheme, the Runge–Kutta discontinuous Galerkin (RKDG) scheme, the weighted essentially non-oscillatory (WENO) scheme, etc. See [3,11,16,20,23,33,36] and the references therein for detailed discussions of the theory.

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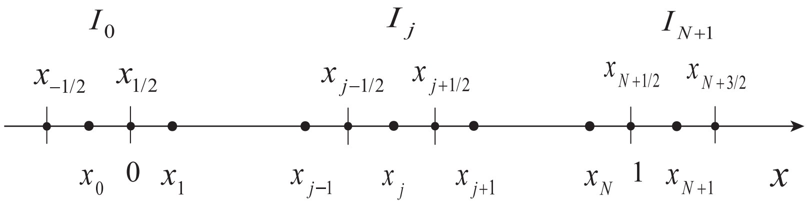

Fig. 1. Cell division for space discretization.

For conservation laws of Eq. (2), the aforementioned higher-order nonlinear schemes can be exploited for the capture of shocks. A straightforward treatment is to regard $\theta(x)$ as being continuous at the cell boundary $x_{j+1/2}$, or view $\theta(x) = \theta(x_{j+1/2})$ as being locally constant around $x_{j+1/2}$, and then directly applies these schemes by taking the numerical flux as $\hat{f}_{j+1/2} = \tilde{f}(u^-_{j+1/2}, u^+_{j+1/2}, \theta(x_{j+1/2}))$, where $\tilde{f}$ is a classical Riemann solver. However, such a treatment was indicated not to be consistent with the steady-state solution or stationary shock of Eq. (2), and oscillations were observed with relatively sharp change in $\theta(x)$ [43–45]. We note that Eq. (2) usually gives a nonconstant steady-state solution $u = u(x)$, other than a trivial or constant solution that is implied in Eq. (1) or by setting $\theta(x)$ as being constant in Eq. (2).

Zhang and Liu [43,44] proposed a so-called δ-mapping algorithm based on a thorough study of the characteristic theory under the scalar form of Eq. (2). The algorithm first assumes an intermediate state/value $\theta_{j+1/2} = \bar{\theta}(\theta_j, \theta_{j+1})$ of $\theta(x)$, which is somehow between the jth and $(j+1)$th cells, and then maps u_j and u_{j+1} onto the intermediate state $\theta_{j+1/2}$. The mapping is based on the fact that the flow $f(u, \theta(x))$ (other than the solution variable u) is constant in a characteristic. With the mapped values $\delta_{j+1/2}u_j$ and $\delta_{j+1/2}u_{j+1}$, the two adjacent solution states are "unified" at a frozen state $\theta_{j+1/2}$, and a classical numerical flux $\tilde{f}(\delta_{j+1/2}u_j, \delta_{j+1/2}u_{j+1}, \theta_{j+1/2})$, e.g., the well known Godonov, Lax–Fridrichs, or Engquist–Osher flux is used to approximate the flux $f(u(x_{j+1/2}, t), \theta(x_{j+1/2}))$ at the cell boundary.

Here, the dependency of numerical fluxes on θ_j and θ_{j+1} (other than $\theta(x_{j+1/2})$) implies that $\theta(x)$ is essentially discontinuous with respect to x in $\theta(x)$, as is the flux $f(u, \theta(x))$. In this regard, Eq. (2) is called conservation laws with discontinuous fluxes in the literature. In fact, system (2) could be standardized by including the following equation: $\theta_t = 0$. Accordingly, θ is viewed as a solution variable and must be taken as being discontinuous in the Riemann problem. We refer the reader to [47] for detailed discussions of the standardized system.

The so called δ-mapping algorithm was further developed for solving Eq. (2), through combination with the RKDG scheme for the LWR model of traffic flow, with the WENO scheme for the elastic wave in heterogeneous media [41] and the multi-class model of traffic flow [48]. These "hybrid" schemes are different from the aforementioned "straightforward treatment" in that $\delta_{j+1/2}u_i$ (other than u_i) were adopted in a classical numerical flux $\tilde{f}$, where i refers to all involved cells for approximating $f(u, \theta)$ at $x = x_{j+1/2}$. These schemes were verified to be consistent with the steady-state flow or stationary shock of Eq. (2). We mention that other schemes, e.g., those developed for Eq. (2) in [2,6,17,18,21,24,38,39], possess the same consistency and thus are able to well resolve the solution profiles despite much differences between their formulations and those in [41,43,44,48].

The present paper proposes a hybrid scheme for solving Eq. (2) by combining the δ-mapping algorithm with the higher-order accurate RKDG scheme. Since the RKDG scheme (as well as TVD scheme) adopts a limiter that suggests nonlinearity or viscosity wherever near a shock, the δ-mapping is also adopted in the limiter to maintain the aforementioned consistency with steady-state solutions or stationary shocks. Precisely, $u_{j\mp1}$ are replaced by $\delta_j u_{j\mp1}$ in the limiter referring to the jth cell, where δ_j corresponds to the θ_j state. Although the discussion succeeds to that in [45], we deal with the system more than the scalar equation, and focus on the multi-class traffic flow [5–10,13,14,27,28,37,40,42,46,47] and nonlinear elasticity in heterogeneous media [2,23–25,41]. The numerical results demonstrate that the scheme is robust in resolving the complex waves in the aforementioned problems, which are comparable with those given by the hybrid scheme that combines δ-mapping and the fifth-order accurate WENO scheme in [41,48], and those in [24].

The remainder of this paper is organized as follows. In Section 2, the RKDG scheme together with its combination with the δ-mapping algorithm for the system of (2) is discussed in general. In Section 3, the consistency with the steady-state solution of Eq. (2) is discussed (Section 3.1); the aforementioned hybrid scheme is implemented with detailed discussions for elastic waves in heterogeneous media (Section 3.2) and for multi-class traffic flow (Section 3.3), respectively; numerical examples are presented in this section. We conclude the paper by Section 4.

2. RKDG method combined with δ-mapping

2.1. General account of DG space discretization

A finite computational interval [0, L] is uniformly divided into cells: $I_j = (x_{j-1/2}, x_{j+1/2})$, with $\Delta_j = x_{j+1/2} - x_{j-1/2}$, and $x_j = (x_{j-1/2} + x_{j+1/2})/2$, $j = 1, \ldots, N$, which is shown by Fig. 1. For Eq. (2) with the initial condition:

$$u(x, 0) = u_0(x), \tag{3}$$