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Non-smooth quadratic centers defined in two arbitrary sectors☆

Isaac A. García*

Departament de Matemàtica, Universitat de Lleida, Avda. Jaume II, 69, Lleida 25001, Spain

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ABSTRACT

In this paper we analyze the center–focus problem of some families of piecewise planar quadratic vector fields on two zones of $\mathbb{R}^2$. The zones we consider are two unbounded sectors defined by an arbitrary angle α and a fixed vertex. We also assume that each vector field share a common weak focus singularity at the vertex of the boundary. We observe how the center variety depends on the angle α.

1. Introduction and statement of the main results

There are many problems arising from branches of science such as physics, mechanics or automatic control which are modeled by non-smooth differential systems (see for example the textbooks [2,9,12]). In this work, we focus on switching planar systems which are defined by two planar smooth vector fields $\mathcal{X}^+$ and $\mathcal{X}^-$ defined on two regions separated by a switching curve. In this context, it makes sense to study the center–focus problem of switching families as the pioneering work [13] begins.

We consider a polynomial non-smooth vector field $\mathcal{X}$ in $\mathbb{R}^2$ with two semi straight lines of discontinuity having end point at the focus–focus singularity. Thus, two zones must be considered and we use the notation $\mathcal{X}=\mathcal{X}^+$ and $\mathcal{X}=\mathcal{X}^-$ in each zone. Locating the singular point at the origin and taking the discontinuity rays to be the positive x-axis Σ_0 and the semi-line $\Sigma_\alpha=\{(x,y)\in\mathbb{R}^2 : x=r\cos(\alpha), y=r\sin\alpha, r\geq 0\}$ with $\alpha\in\mathbb{S}^1=[0,2\pi)$ and end point at the origin, the non-smooth family adopts the form

$$(\dot{x},\dot{y})=\begin{cases}(-y+P^+(x,y;\lambda), x+Q^+(x,y;\lambda)) & \text{if} \quad (x,y)\in S_\alpha^+,\\ (-y+P^-(x,y;\lambda), x+Q^-(x,y;\lambda)) & \text{if} \quad (x,y)\in S_\alpha^-,\end{cases} \tag{1}$$

where $S_\alpha^\pm$ are the two open unbounded sectors with boundary $\Sigma_0\cup\Sigma_\alpha$ such that $S_\alpha^+\cup S_\alpha^-\cup\Sigma_0\cup\Sigma_\alpha=\mathbb{R}^2$ and $S_\alpha^+\cap S_\alpha^-=\emptyset$. Here, λ denotes the vector whose components are the real parameters of the family.

The center–focus problem at the origin of (1) has mainly been considered in the literature when the switching family has one switching line, usually the x-axis. These special switching systems belong to just the particular case $\alpha=\pi$ and the center–focus problem for them has been recently analyzed in several papers [3–5,7,11], specifically in the quadratic case but the general problem remains still open. In these papers the multiple Hopf bifurcations from a focus are also studied and,

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* Tel.: +34 691683386.
E-mail address: garcia@matematica.udl.cat

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due to non-smoothness, more small amplitude limit cycles are created than in the smooth case for a fixed degree of the polynomials $P^{\pm}$ and $Q^{\pm}$. For example, in the quadratic case with $P^- = Q^- \equiv 0$ the work [7] shows that at least 4 limit cycles can bifurcate from the weak focus at the origin. Next Gasull and Torregrosa [11] and later Chen et al. [4] found that 5 limit cycles can bifurcate from a weak focus in a particular case of the general quadratic family called switching Bautin family, see the forthcoming Definition 2. It is worth to emphasize here that, in contrast, the bifurcation of small amplitude limit cycles from the center of a non-smooth family with a switching line is less studied with the exception of [3,4]. Thus few results are known in these Hopf bifurcations from a center: 8 limit cycles are found in switching Bautin systems in both [3,4] whereas 9 limit cycles are created in [3] from the center of a quadratic switching family not belonging to the switching Bautin family.

In the general quadratic case, the right hand side of (1) has arbitrary homogeneous polynomials $P^{\pm}$ and $Q^{\pm}$ in x and y which can be taken, without loss of generality, as

$$\begin{aligned} P^+(x,y;\lambda) &= -A_3x^2 + (2A_2+A_5)xy + A_6y^2,\\ Q^+(x,y;\lambda) &= A_2x^2 + (2A_3+A_4)xy + (A_1-A_2)y^2,\\ P^-(x,y;\lambda) &= -B_3x^2 + (2B_2+B_5)xy + B_6y^2,\\ Q^-(x,y;\lambda) &= B_2x^2 + (2B_3+B_4)xy + (B_1-B_2)y^2, \end{aligned}$$

hence $\lambda = (A_1, A_2, A_3, A_4, A_5, A_6, B_1, B_2, B_3, B_4, B_5, B_6) \in \mathbb{R}^{12}$.

Remark 1. The smooth quadratic family $\dot{x} = -y + P^+(x,y;\lambda)$, $\dot{y} = x + Q^+(x,y;\lambda)$ can be always written (after a rotation in the phase plane) in the called Bautin form, that is, with $A_1 = 0$. In this case, after [1], it is well known that the origin is a center if and only if one of the following four conditions is fulfilled:

(a) $A_4 = A_5 = 0$;
(b) $A_3 = A_6$;
(c) $A_5 = A_4 + 5(A_3 - A_6) = A_3A_6 - 2A_6^2 - A_2^2 = 0$;
(d) $A_2 = A_5 = 0$.

Definition 2. We say that the non-smooth quadratic family (1) is in Bautin form if $A_1 = B_1 = 0$, that is, both $\mathcal{X}^+$ and $\mathcal{X}^-$ are in Bautin form.

We are unable to study the center problem in the full quadratic family, so we only analyze some subfamilies. First, we define persistent center.

Definition 3. The origin is called a *persistent* center of system (1) with $\lambda = \lambda^*$ if it is a center for all $\alpha \in \mathbb{S}^1$.

Remark 4. It is worth to emphasize that our definition of persistent center in non-smooth systems is different to the definition of persistent center that appears in the literature (see [6]) for the smooth case. In [6], the origin of a complex system $\dot{z} = iz + F(z,\bar{z};\lambda)$ with $z = x + iy \in \mathbb{C}$ is said to be a persistent center when it is a center of $\dot{z} = iz + \lambda F(z,\bar{z};\lambda)$ for all $\lambda \in \mathbb{C}$.

A *trivial* persistent center at the origin of (1) takes place when $\mathcal{X}^+ = \Lambda\,\mathcal{X}^-$ where $\mathcal{X}^+$ has a center at the origin and Λ is a real analytic function defined near the origin of $\mathbb{R}^2$ with $\Lambda(0,0) \neq 0$. Equivalently, we characterize such persistent centers when both vector fields $\mathcal{X}^{\pm}$ share a common analytic first integral in a neighborhood of it.

An interesting question to solve is: may non-trivial centers exist? We will prove that the answer is no for the quadratic case.

Theorem 5. *The origin of any non-smooth planar quadratic family (1) is a persistent center if and only if it is trivial.*

Now we assume that $\mathcal{X}^-$ is linear and we will solve the quadratic center problem in some cases (actually when either $A_2 = A_3 = 0$ or $A_5 = 0$, see Theorems 6 and 7). First we observe that, since the vector field $\mathcal{X}^- = -y\partial_x + x\partial_y$ is invariant under rotations, we can assume without loss of generality $A_1 = 0$, see Lemma 12. Therefore the non-smooth family we will study is in the Bautin form

$$(\dot{x},\dot{y}) = \begin{cases} (-y + P^+(x,y;\lambda), x + Q^+(x,y;\lambda)) & \text{if} \quad (x,y) \in S_\alpha^+, \\ (-y, x) & \text{if} \quad (x,y) \in S_\alpha^-, \end{cases} \tag{2}$$

with

$$\begin{aligned} P^+(x,y;\lambda) &= -A_3x^2 + (2A_2+A_5)xy + A_6y^2,\\ Q^+(x,y;\lambda) &= A_2x^2 + (2A_3+A_4)xy - A_2y^2. \end{aligned}$$

The non-smooth quadratic family (2) with $\alpha = \pi$ was considered in [7,12] whereas in [11] the center problem at the origin was solved again for $\alpha = \pi$.