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Improving the convergence order of the regularization method for Fredholm integral equations of the second kind

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ABSTRACT

We build a numerical approximation method, for Fredholm integral equation solution of the second type. This method is based on the regularization by convolution and Fourier series expansion. It provides a better convergence order.

1. Introduction

The calculation of the objects related to linear operators in infinite dimension is usually impossible. So we find that we are forced to build numerical approximations.

We shall construct an approximate solution of Fredholm integral equation of the second kind:

$$u(s) = Ku(s) + f(s), \quad s \in [0, 1],$$

where K, is an integral operator with a weakly singular kernel. This kind of operators is very interesting, as you can see in [1–3]. The originality of our work is summed up in the improvement of the convergence order obtained in previous work. We use a method studied in [4–6].

This method is divided into two parts: in the first part, we regularize the kernel using convolution. In the second part, we construct an operator with a degenerate kernel using truncated Fourier series.

2. Position of the problem

Let $\mathcal{B}$ be a Banach space of all bounded linear operators defined on $L^1([0, 1], \mathbb{C})$ to itself. The norm of $\mathcal{B}$ is given by :

$$\forall A \in \mathcal{B}, \quad \|A\|_{\mathcal{B}} = \sup_{\|x\|_{L^1([0,1],\mathbb{C})}=1} \|Ax\|_{L^1([0,1],\mathbb{C})}.$$

Let $g \in L^1([0, 1], \mathbb{R})$ be a decreasing positive function, continuous over $]0, 1]$ and

$$\lim_{t\to 0^+} g(t) = +\infty.$$

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Let K be a linear operator defined on $L^1([0,1],\mathbb{C})$ to itself by :

$\forall x \in L^1([0,1],\mathbb{C}), \quad Kx(s) := \int_0^1 g(|s-t|)x(t)dt.$

It is easy to establish that $K \in \mathcal{B}$ and $\|K\|_{\mathcal{B}} \leq 2\|g\|_{L^1([0,1],\mathbb{R})}$. The operator K is an integral operator with a weakly singular kernel. The singularity comes from the condition :

$$\lim_{t\to 0^+} g(t) = +\infty.$$

But it remains weak because,

$$\int_0^1 g(t)dt < +\infty.$$

We recall that any integral operator with a continuous kernel is compact [3]. This property of integral operators allows us to establish the following result:

Theorem 1. *K is a compact operator.*

Proof. We have $C^0([0,1],\mathbb{R})$ is dense in $L^1([0,1],\mathbb{R})$, then there is a sequence $(g_n)_{n\geq 1}$ in $C^0([0,1],\mathbb{R})$ converges to g with respect to the norm $L^1([0,1],\mathbb{R})$. Let,

$$\forall x \in L^1([0,1],\mathbb{C}), \quad K_n x(s) := \int_0^1 g_n(|s-t|)x(t)dt, \quad n \geq 1.$$

Then for all $n \geq 1$, $K_n \in \mathcal{B}$. Each operator K_n is compact on $L^1([0,1],\mathbb{C})$, because its kernel is continuous. But,

$$\|K_n - K\|_{\mathcal{B}} \leq 2\|g_n - g\|_{L^1([0,1],\mathbb{C})}.$$

Thus, the sequences operator $\{K_n\}_{n\in\mathbb{N}}$ converges in norm to K and K is compact. □

Let I be the identity operator. Assuming that:
1 is not an eignvalue of K.
So $I - K : L^1([0,1],\mathbb{C}) \to L^1([0,1],\mathbb{C})$ is invertible and its inverse is bounded, and for any $f \in L^1([0,1],\mathbb{C})$, the equation:

$$u(s) - Ku(s) = f(s); \quad s \in [0,1],$$

has a unique solution $u \in L^1([0,1],\mathbb{C})$.

Our work is to construct a numerical approximation of u. We regularize the kernel using a convolution, then we replace the kernel obtained by a Fourier series truncated at some order. We get an operator with finite rank kernel. We solve the linear system corresponding to approach u.

3. Regularization

To remove the constraint $\lim_{x\to 0} g(x) = +\infty$, we will regularize the kernel by convolution. $\forall x \in L^1([0,1],\mathbb{C})$, Kx is to extend in this way:

$$Kx(s) = Kx(2-s), \ s \in [1,2].$$

Then 2-periodical on $\mathbb{R}$.
For $\varepsilon > 0$, we define K_ε by:

$$K_\varepsilon x = J_\varepsilon * Kx, \quad \forall x \in L^1([0,1],\mathbb{C}),$$

where, J is a function defined on $\mathbb{R}$ satisfying the following properties :

$$\begin{cases} J \text{ is even,} \\ J(x) \geq 0 \ \forall x \in \mathbb{R}, \\ J \in C^m(\mathbb{R},\mathbb{R}), \quad m > 1 \\ \text{Supp}(J) \subset [-1,1], \\ \int_{-1}^1 J(\tau)d\tau = 1, \end{cases}$$

and for $\varepsilon > 0$,

$$J_\varepsilon(\tau) := \varepsilon^{-1} J\left(\varepsilon^{-1}\tau\right).$$

Indeed, for all $x \in L^1([0,1],\mathbb{C})$, $Kx \in L^1([0,1],\mathbb{C})$. Then, $K_\varepsilon x \in L^1([0,1],\mathbb{C})$.

$\forall t \in [0,1]$, we note : $g_t(s) = g(|s-t|)$. $g_t(s)$ is extended as follows $g_t(s) = g_t(2-s)$ for all $s \in [1,2]$. Then 2-periodical on $\mathbb{R}$.

$\forall \varepsilon > 0$, we define:

$$g_\varepsilon(s,t) = J_\varepsilon * g_t(s).$$