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Nondegeneracy of optimality conditions in control problems for a radiative–conductive heat transfer model

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ABSTRACT

A boundary control problem for a nonlinear steady-state heat transfer model accounting for heat radiation effects is considered. The problem consists in the minimization of a cost functional by controlling the reflection properties of the boundary. The solvability of the control problem is proven, an optimality system is derived, and the nondegeneracy of optimality conditions is established. The results of numerical simulations are presented.

1. Introduction

The interest in studying problems of complex heat transfer (where the radiative, convective, and conductive contributions are simultaneously taken into account) is motivated by their importance for many engineering applications. The common feature of such processes is the radiative heat transfer dominating at high temperatures. The radiative heat transfer equation (RTE) is a first order integro-differential equation governing the radiation intensity. The radiation traveling along a path is attenuated as a result of absorption and scattering. On the other hand, the radiation is created according to the Boltzmann emission law. The precise derivation and analysis of such models can be found in the monograph [1].

Solutions to the RTE can be represented in the form of the Neumann series whose terms are powers of an integral operator applied to a certain start function. The terms can be calculated using a Monte Carlo method, which may be interpreted as tracking the history of energy bundles from emission to absorption at a surface or within a participating medium. The method assumes that the bundles start from random points, propagate in random directions, and show the energy exchange due to random scattering (see e.g. [2]).

A way to avoid solving the integro-differential RTE is the expansion of the local intensity in terms of spherical harmonics, with the truncation after N terms in the series, and the substitution of it into the moments of the RTE (see e.g. [1]). This approach leads to the so-called P_N approximations, where N is the order of the expansion. Especially interesting is the P_1 approximation (diffusion approximation) because it does not require high computational efforts. Using the diffusion model instead of the integro-differential RTE is popular in various applications. For e.g., in [3], the image reconstruction of a highly scattering inhomogeneous medium is studied on the basis of the diffusion model. The mathematical justification of using the diffusion approximation for modeling radiative transfer in biological tissues can be found in [4]. As the diffuse

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approximation is the core of our paper, it is worth to mention the work [5] showing that the diffusion approximation of the radiative–conductive heat transfer describes the behavior of the temperature properly. Thus, the diffusion approximation can successfully be applied to various heat transfer problems if a very high accuracy is not required.

Optimal control problems of complex heat transfer draw the interest of researchers in various application areas, e.g. glass manufacturing [6–8], laser thermotherapy [9], design of cooling systems [10,11], etc. A considerable number of works is devoted to control problems related to evolutionary systems describing radiative heat transfer (see e.g. [6–10,12–14]). In the above-mentioned works, the transfer of radiation is described by an integro-differential equation or by its approximations. The temperature field is simulated by the conventional evolutionary heat transfer equation with additional source terms accounting for the contribution of radiation.

Investigation of optimal boundary control problems for steady-state systems of complex heat transfer is far from being completed. For example, in concern with necessary optimality conditions, the proof of the epimorphism property of the derivative of the control-to-state mapping is significantly more complicated than that in the evolutionary case. In this connection, it is worth to mention the work [11], where an optimal boundary multiplicative control problem for a steady-state complex heat transfer model is considered. The problem is formulated as the maximization of the energy outflow from the model domain by controlling reflection properties of the boundary. The solvability of this problem is proven based on new a priori estimates for solutions of the model equations. The main result of this work is the proof of an analogue of the bang-bang principle arising in control theory for ordinary differential equations. It should be noticed that nondegeneracy of necessary optimality conditions obtained was not proven in general.

The present paper can be considered as an enhancement of the above-discussed one. The minimization of a general cost functional by controlling the boundary reflectivity is investigated. This includes e.g. obtaining a desired distribution of the temperature or the radiation intensity in a part of the model domain. The solvability of the optimal control problem is proven and necessary optimality conditions are derived. Additionally, the nondegeneracy of the optimality conditions is proven without any smallness assumptions on the problem data. An iterative algorithm for finding optimal controls is proposed. The proven bang-bang structure of optimal controls provides a rapid convergence of the iterative procedure.

2. Problem formulation

The following steady-state normalized diffusion, P_1, model (see [1,15–17]) describing radiative–conductive heat transfer in a bounded domain $G \subset \mathbb{R}^3$ is under consideration. The model equations read:

$$-a\Delta\theta + b\kappa_a(|\theta|\theta^3 - \varphi) = 0, \quad -\alpha\Delta\varphi + \kappa_a(\varphi - |\theta|\theta^3) = 0, \tag{1}$$

$$a\partial_n\theta + \gamma(\theta - \theta_b)|_\Gamma = 0, \quad \alpha\partial_n\varphi + u(\varphi - \theta_b^4)|_\Gamma = 0. \tag{2}$$

Here, θ is the normalized temperature, φ the normalized radiation intensity averaged over all directions, κ_a the absorption coefficient, and θ_b and γ are given non-negative functions describing the normalized external temperature and the normalized overall heat transfer coefficient, respectively. The other parameters are given by the formulae

$$a = \frac{k}{\rho c_v}, \quad b = \frac{4\sigma n^2 T_{max}^3}{\rho c_v}, \quad \alpha = \frac{1}{3\kappa - A\kappa_s},$$

where k is the thermal conductivity, c_v the specific heat capacity, ρ the density, σ the Stefan–Boltzmann constant, n the refractive index, T_{max} the maximum temperature in the unnormalized model, $\kappa := \kappa_s + \kappa_a$ the extinction coefficient (total attenuation factor), and κ_s the scattering coefficient. The coefficient $A \in [-1, 1]$ describes the anisotropy of scattering. The symbol ∂_n denotes the derivative in the outward normal direction $\mathbf{n}$ on the boundary $\Gamma := \partial G$.

The function u represents the boundary control. It is assumed that the control is constrained by the inequalities

$$u_1 \le u \le u_2, \tag{3}$$

where u_1 and u_2 are given non-negative functions.

The problem of optimal control consists in the determination of functions u, θ, and φ satisfying conditions (1)–(3) and minimizing a cost functional $J(\theta, \varphi)$. In particular, the functional J can describe the L^2-deviation of the temperature and radiation field from prescribed distributions, θ_d and φ_d. That is,

$$J(\theta, \varphi) = a_\theta \|\theta - \theta_d\|^2_{L^2(G)} + a_\varphi \|\varphi - \varphi_d\|^2_{L^2(G)},$$

where a_θ and a_φ are nonnegative weights.

3. Formalization of the optimal control problem

Suppose that the model data satisfy the following conditions:

(i) $\theta_b, \gamma, u_1, u_2 \in L^\infty(\Gamma); \theta_b > 0, \gamma \ge \gamma_0 > 0, u_{1,2} \ge u_0 > 0$.
(ii) The cost functional $J : H^1(G) \times H^1(G) \to \mathbb{R}$ is weakly lower semicontinuous.
(iii) The functional J is Frechet differentiable.