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Bistability and multistability in opinion dynamics models[☆]

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ABSTRACT

In this paper, we analyze a few opinion formation or election game dynamics models. For a simple model with two subpopulations with opposing opinions *A* and *B*, if the fraction of committed believers of opinion *A* ("*A* zealots") is less than a critical value, then the system has two bistable equilibrium points. When the fraction is equal to the critical value, the system undergoes a saddle-node bifurcation. When the fraction is larger than the critical value, a boundary equilibrium point is globally asymptotically stable, suggesting that the entire population reaches a consensus on *A*. We find a similar bistability property in the model in which both opinions have their own zealots. We also extend the model to include multiple competitors and show the dynamical behavior of multistability. The more competitors subpopulation *A* has, the fewer zealots opinion *A* needs to obtain consensus.

1. Introduction

Sociophysics studying the social and political behavior with modeling and simulation tools has been a research area with growing interest to physicists and mathematicians [5,6,19]. Opinion dynamics is one of the most active research subjects of sociophysics [4,7–9,16–18,20]. Motivated by the study of language competition dynamics, Castelló et al. used a modification of a voter model to study the ordering dynamics with two non-excluding options [1]. Colaiori and Castellano [3] studied the influence of both media and peer pressure on opinion dynamics. Built upon other work on opinion dynamics [21], Marvel et al. [12] modified a simple model of opinion spreading and found that only one of the modifications can significantly expand the moderate subpopulation. Nyczka et al. [14] studied a generalized version of the Sznajd model [11] and showed that opinion dynamics can be understood as a movement of a public opinion in a symmetric bistable effective potential. Inspired by the work in [14], we will analyze the simple model in the ref. [12] and its generalizations. There are other works studying opinion spreading models, such as [2,4,10,13,15,17]. The roles of informed agents and social power on opinion formation have been evaluated in [22–24]. Competition and cooperation in the study of social behavior have also been addressed by the evolutionary game theory [25,26].

The simple model in [12] studied the dynamics of three subpopulations holding three opinion states: an extreme opinion *A*, a conceptually opposing opinion *B*, and neither *A* nor *B* (the moderates). The model diagram is shown in Fig. 1(a) and the

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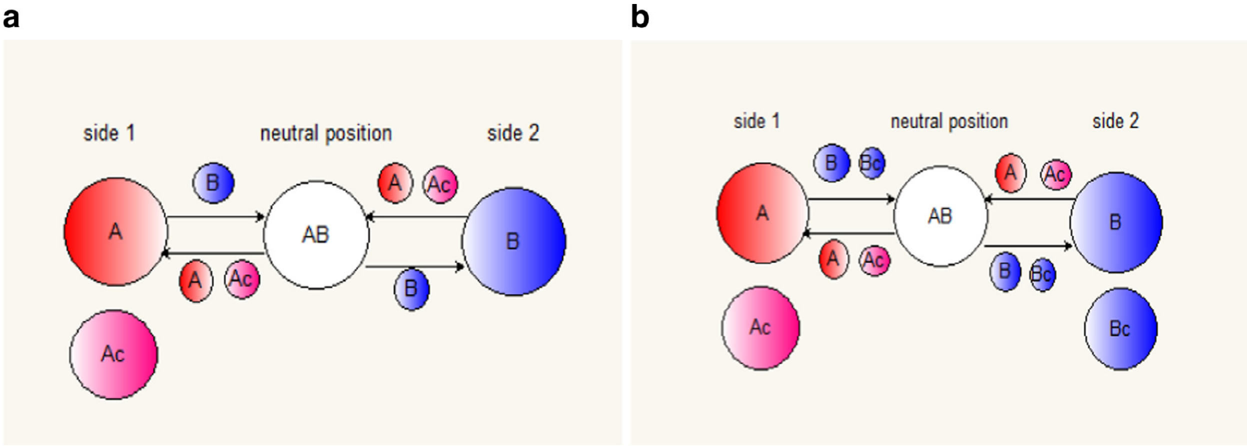

Fig. 1. (a) Diagram of model (1.1), adapted from the ref. [12]. (b) Diagram of model (3.1). *AB* represents the subpopulation that holds neither opinion *A* nor *B* (the moderates). The subscript *c* represents the subpopulation that holds the opinion indefinitely. A speaker can convert a listener from one subpopulation to another.

model is described by the following ordinary differential equations.

$$\begin{cases} \dot{n}_A = (p + n_A)n_{AB} - n_A n_B, \\ \dot{n}_B = n_B n_{AB} - (p + n_A)n_B, \\ \dot{n}_{AB} = n_A n_B + (p + n_A)n_B - (p + n_A)n_{AB} - n_B n_{AB}. \end{cases} \tag{1.1}$$

In the model, n_A, n_B and n_{AB} denote the proportions of those who currently hold opinion *A*, opinion *B*, and neither *A* nor *B*, respectively. A speaker can convert a listener from one subpopulation to another (for example, *B* can convert *A* to an undecided opinion state *AB* and *AB* can be converted by *A* to *A*, see the diagram in Fig. 1a). The model also considers people who hold opinion *A* indefinitely and cannot be influenced by others (the so-called "*A* zealots", denoted by A_c). Let p ($0 \le p \le 1$) be the proportion of *A* zealots in the total population.

Because $n_{AB} = 1 - p - n_A - n_B$, we can convert model (1.1) to the following system.

$$\begin{cases} \dot{n}_A = p(1-p) + (1-2p)n_A - pn_B - n_A^2 - 2n_A n_B, \\ \dot{n}_B = (1-2p)n_B - n_B^2 - 2n_A n_B. \end{cases} \tag{1.2}$$

In this paper, we will analyze model (1.2) and its generalizations. Bistability and multistability will be shown to appear in these opinion dynamics models.

2. Model analysis

2.1. Positivity and equilibria

Let $\mathscr{D}^* = \left\{(n_A, n_B, n_{AB}) \in \mathbb{R}_+^3 | n_A + n_B + n_{AB} = 1 - p\right\}$ and $\mathscr{D} = \left\{(n_A, n_B) \in \mathbb{R}_+^2 | n_A + n_B \le 1 - p\right\}$. We will show that system (1.1) or (1.2) is well-posed.

Theorem 2.1. *All solutions of system (1.1) are eventually confined in the compact subset* $\mathscr{D}^*$. *All solutions of system (1.2) are nonnegative. Moreover,* $\mathscr{D}$ *is a positive invariant for (1.2).*

Proof. For any positive initial conditions (n_A^0, n_B^0, n_{AB}^0), we assume that t_1 is the first time such that $n_A(t_1)n_B(t_1)n_{AB}(t_1) = 0$. Thus, there are three possible cases: (i) $n_A(t_1) = 0, n_B(t) \ge 0, n_{AB}(t) \ge 0$; (ii) $n_B(t_1) = 0, n_A(t) \ge 0, n_{AB}(t) \ge 0$; (iii) $n_{AB}(t_1) = 0, n_A(t) \ge 0, n_B(t) \ge 0$, for $t \in [0, t_1]$.

For case (i), it is obvious that $\dot{n}_A(t_1) < 0$. On the other hand, from the first equation of system (1.1), we have $\dot{n}_A(t_1) = pn_{AB}(t_1) \ge 0$, which is a contradiction.

For case (ii), from the second equation of system (1.1) we have $\dot{n}_B(t) \ge -(p + n_A)n_B$ for $t_2 \in [0, t_1]$. Thus $n_B(t_1) \ge n_B^0 e^{-(p+n_A)t} > 0$, which is contradiction with $n_B(t_1) = 0$.

For case (iii), it is clear that $\dot{n}_{AB}(t_1) < 0$. On the other hand, from the third equation of system (1.1) we have $\dot{n}_{AB}(t_1) = pn_B(t_1) + 2n_A(t_1)n_B(t_1) \ge 0$, which is a contradiction.

Thus we showed that all the solutions of system (1.1) with positive initial data are positive. Let (n_A, n_B, n_{AB}) be any nonnegative solution and $N = n_A + n_B + n_{AB}$. The time derivative along a solution of (1.1) is $\dot{N} = 0$. Thus, we know that $n_A + n_B + n_{AB}$ is a constant. Thus, the set $\mathscr{D}^*$ is a positive invariant with respect to (1.1). Since $n_{AB} = 1 - p - n_A - n_B$ is nonnegative, we have $n_A + n_B \le 1 - p$. Therefore, $\mathscr{D}$ is a positive invariant for (1.2).

To examine the existence of equilibrium points, we let $\Delta = 4p^2 - 8p + 1$. □