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Upper bounds for some graph energies

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ABSTRACT

A general inequality for non-negative real numbers is proven. Based on it, upper bounds for (ordinary) graph energy, minimum dominating energy, minimum covering energy, Laplacian-energy-like invariant, Laplacian energy, Randić energy, and incidence energy are obtained.

1. Introduction

Let $G = (V, E)$, $V = \{v_1, v_2, \ldots, v_n\}$, be a simple graph with $n = |V|$ vertices and $m = |E|$ edges. If the vertices v_i and v_j are adjacent, then we write $v_i \sim v_j$. Denote by d_i the degree (number of first neighbors) of the vertex v_i, and assume that $d_1 \geq d_2 \geq \cdots \geq d_n$. Some well known properties of the vertex degrees are [7]:

$$\sum_{i=1}^{n} d_i = 2m \qquad \text{and} \qquad \sum_{i=1}^{n} d_i^2 = \sum_{v_i \sim v_j} (d_i + d_j) = M_1$$

where M_1 is the first Zagreb index [17,21].

The adjacency matrix $\mathbf{A} = (a_{ij})$ of the graph G is defined as

$$a_{ij} = \begin{cases} 1 & \text{if } v_i \sim v_j \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of $\mathbf{A}$, denoted by $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, are referred to as the (ordinary) eigenvalues of the graph G [7]. Some of their well known properties are:

$$\sum_{i=1}^{n} \lambda_i = 0 \qquad \text{and} \qquad \sum_{i=1}^{n} \lambda_i^2 = \sum_{i=1}^{n} d_i = 2m.$$

Denote by $|\lambda_1^*| \geq |\lambda_2^*| \geq \cdots \geq |\lambda_n^*|$ a non-increasing sequence of absolute values of the eigenvalues of G. The graph invariant $E = E(G)$, called energy of G, is defined to be the sum of the absolute values of the eigenvalues of G [20,27], i.e.,

$$E = E(G) = \sum_{i=1}^{n} |\lambda_i| = \sum_{i=1}^{n} |\lambda_i^*|.$$

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A subset $\mathcal{D}$ of V is said to be a dominating set of G if every vertex of $V\setminus\mathcal{D}$ is adjacent to some vertex in $\mathcal{D}$. Any dominating set with minimum cardinality is called a minimum dominating set. Let $\mathcal{D}$ be a minimum dominating set of the graph G. The minimum dominating adjacency matrix of G, denoted by $\mathbf{A}_{\mathcal{D}}=(a_{ij}^{\mathcal{D}})$, is the $n\times n$ matrix defined as

$$a_{ij}^{\mathcal{D}}=\begin{cases}1 & \text{if } v_i\sim v_j\\ 1 & \text{if } i=j,\ v_i\in\mathcal{D}\\ 0 & \text{otherwise.}\end{cases}$$

The minimum dominating eigenvalues of the graph G, $\alpha_1\ge\alpha_2\ge\cdots\ge\alpha_n$, are the eigenvalues of $\mathbf{A}_{\mathcal{D}}$. The following equalities are valid for them [24]:

$$\sum_{i=1}^{n}\alpha_i=|\mathcal{D}|\qquad\text{and}\qquad\sum_{i=1}^{n}\alpha_i^2=2m+|\mathcal{D}|.$$

Let $|\alpha_1^*|\ge|\alpha_2^*|\ge\cdots\ge|\alpha_n^*|$, $\alpha_1=|\alpha_1|=|\alpha_1^*|$, be a non-increasing sequence of absolute values of the minimum dominating eigenvalues of G. The minimum dominating energy of the graph G, $E_{\mathcal{D}}=E_{\mathcal{D}}(G)$, is defined as [24,38]

$$E_{\mathcal{D}}=E_{\mathcal{D}}(G)=\sum_{i=1}^{n}|\alpha_i|=\sum_{i=1}^{n}|\alpha_i^*|.$$

A subset C of V, $C\subset V$, is called a covering set of G if every edge of G is incident to at least one vertex of G. Any covering set with minimum cardinality is called a minimum covering set. Let C be a minimum covering set of the graph G. The minimum covering matrix of G is the matrix defined by $\mathbf{A}_C=(a_{ij}^C)$, where

$$a_{ij}^{C}=\begin{cases}1 & v_i\sim v_j\\ 1 & i=j, v_i\in C\\ 0 & \text{otherwise.}\end{cases}$$

The minimum covering eigenvalues of G, denoted by $\beta_1\ge\beta_2\ge\cdots\ge\beta_n$, are the eigenvalues of $\mathbf{A}_C$. The following equalities are valid for β_i, $i=1,2,\ldots,n$ [1]

$$\sum_{i=1}^{n}\beta_i=|C|\qquad\text{and}\qquad\sum_{i=1}^{n}\beta_i^2=2m+|C|.$$

Let $|\beta_1^*|\ge|\beta_2^*|\ge\cdots\ge|\beta_n^*|$, $\beta_1=|\beta_1|=|\beta_1^*|$, be a non-increasing sequence of absolute values of the minimum covering eigenvalues of G. The minimum covering energy of the graph G, $E_C=E_C(G)$, is defined as [1]

$$E_C=E_C(G)=\sum_{i=1}^{n}|\beta_i|=\sum_{i=1}^{n}|\beta_i^*|.$$

Let $\mathbf{D}=\mathrm{diag}\{d_1,d_2,\ldots,d_n\}$. Then $\mathbf{L}=\mathbf{D}-\mathbf{A}$ is the Laplacian matrix of G. The eigenvalues of $\mathbf{L}$, denoted by $\mu_1\ge\mu_2\ge\cdots\ge\mu_n=0$, are the Laplacian eigenvalues of the graph G. Some of their well known properties are [15]:

$$\sum_{i=1}^{n-1}\mu_i=\sum_{i=1}^{n}d_i=2m\qquad\text{and}\qquad\sum_{i=1}^{n-1}\mu_i^2=\sum_{i=1}^{n}d_i^2+\sum_{i=1}^{n}d_i=M_1+2m.$$

Let $\gamma_i=\mu_i-\frac{2m}{n}$, $i=1,2,\ldots,n$, and $|\gamma_1^*|\ge|\gamma_2^*|\ge\cdots\ge|\gamma_n^*|$, $|\gamma_1^*|=\frac{2m}{n}$, be a sequence of non-increasing values of absolute values of γ_i. The Laplacian energy of G is defined as [22]

$$LE=LE(G)=\sum_{i=1}^{n}\left|\mu_i-\frac{2m}{n}\right|=\sum_{i=1}^{n}|\gamma_i^*|.$$

Since for several reasons the concept of Laplacian energy was not fully satisfactory, a seemingly different quantity, named Laplacian-energy-like invariant, *LEL*, has been put forward [30], defined as

$$LEL=LEL(G)=\sum_{i=1}^{n-1}\sqrt{\mu_i}.$$

The matrix $\mathbf{L}^+=\mathbf{D}+\mathbf{A}$ is the signless Laplacian matrix of the graph G. Its eigenvalues, denoted by $\mu_1^+\ge\mu_2^+\ge\cdots\ge\mu_n^+\ge 0$, are referred to as the signless Laplacian eigenvalues of G [8]. Some of their well known properties are

$$\sum_{i=1}^{n}\mu_i^+=2m\qquad\text{and}\qquad\sum_{i=1}^{n}\left(\mu_i^+\right)^2=M_1+2m.$$