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Artificial boundary conditions for the Burgers equation on the plane

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ABSTRACT

The numerical solution of the initial value problem for the two-dimensional Burgers equation on the whole plane is considered. Usual techniques, like finite difference methods and finite element methods cannot be directly applied for the solution of this problem, because the corresponding domain is unbounded. We propose a new method to overcome this difficulty. The efficiency of the proposed method is tested by several numerical examples.

1. Introduction

The Burgers equation is a mathematical model employed in a large number of applications. It was introduced by Bateman [1] and it was later studied by Burgers [2,3], as a mathematical model for turbulence in fluid dynamics. However, after these preliminary studies, it has been considered in various areas of applied mathematics [4], such as for examples: gas dynamics, waves in shallow water, hydromagnetic waves in cold plasma, ion-acoustic waves in cold plasma [5]; modeling of traffic flow [6,7]; dynamics of soil water [8]; shock formation in inelastic gases [9]; cosmology [10]; turbulence in fluid dynamics [11,12].

Besides its effective modeling features, it can be used to set-up computational methods in order to deal with more difficult problems as the Navier–Stokes equation, whose analytical properties and approximation techniques are only partially known. The three-dimensional incompressible Navier–Stokes equation is one of the main open problems in Mathematics; the Clay Mathematics Institute [13] has identified the existence, smoothness and breakdown of the Navier–Stokes solution as one of the millennium problems.

In Burgers equation, discontinuities may appear in finite time, even if the initial condition is smooth [14,15]. These discontinuities give rise to the phenomenon of shock waves with important applications in physics [16]. So that, Burgers equation is a proper model to test numerical algorithms for the simulation in flows with severe gradients or shocks, and it provides a good training model to assess stability and efficiency of computational methods.

Various numerical techniques have been proposed to solve Burgers equation, such as: finite difference methods [17–21]; finite element methods [18,22,23]; Adomian decomposition method [24–27]; integral equations methods [14,23,28,29]; cellular automata methods [30,31]. Finite difference methods and finite element methods are the most general and effective techniques for the numerical solution of differential equations. However, these methods are able to solve differential equations on bounded domains. In order to deal with unbounded domains, suitable numerical techniques must be considered. In [23,32] two different artificial boundary methods are proposed. The general idea of artificial boundary methods is to reduce

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an initial value problem on an unbounded domain to an equivalent problem on a bounded artificial domain by introducing suitable conditions on the artificial boundary. This kind of methods, after the trailblazing paper [33] for the wave equation, has been proposed for the Helmholtz equation [34], for diffusion equations [35], for Maxwell equations [36], and for Navier–Stokes equation [37,38].

In the present paper, we develop the idea described in [23], where the integral formulation of the Burgers equation on the Fourier transform space is used to compute the artificial boundary condition, and the Galerkin method is used to solve the resulting initial-boundary value problem. In particular, an efficient method for the computation of these boundary conditions is proposed; this method is based on the numerical solution of the above mentioned integral equation by using FFT algorithm.

The main idea of the proposed method is to follow the solution of the problem in the original space as well as in the Fourier transform space. In particular, the numerical solutions of these two formulations are computed by using a truncation of the original space and of the conjugate space, respectively, so these solutions provide a complementary information of the original problem in the unbounded domain. This is a simple idea, but, from our experience, it produces an effective technique. Moreover it can be easily generalized to deal with Navier–Stokes equation, by using the corresponding integral formulation on the Fourier transform space [39]. A numerical experiment is used to test the efficiency and the accuracy of the proposed method on several numerical examples.

In Section 2, the integral formulation of the Burgers equation is derived and the FFT algorithm for its solution is given. In Section 3, we illustrate the Galerkin algorithm used for the numerical solution of Burgers equation on a bounded domain. In Section 4 we give the artificial boundary condition method proposed to solve Burgers equation on the whole plane. In Section 5 some numerical experiments with the proposed method are presented. In Section 6 conclusions and future developments are given.

2. Integral equation

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{R}$ the set of real numbers, $\mathbb{R}^+$ the set of positive real numbers, and $\mathbb{C}$ the set of complex numbers. We denote with ι the imaginary unit. Let $d \in \mathbb{N}$, we denote with $\mathbb{R}^d$, $\mathbb{C}^d$ the d-dimensional real Euclidean space and the d-dimensional complex Euclidean space, respectively. Let $\underline{x}$, $\underline{y} \in \mathbb{R}^d$ be column vectors, we denote with $\underline{x}^T$ the transpose of $\underline{x}$, with $\underline{x}^T \underline{y}$ the scalar product of $\underline{x}$ and $\underline{y}$, and with $|\underline{x}|$ the Euclidean norm of $\underline{x}$. Let $\mathcal{L}^2$ be the space of functions that are square integrable, and $\mathcal{L}^1$ be the space of functions whose absolute value is integrable.

Let $\underline{f} : \mathbb{R}^d \to \mathbb{C}^k$, $\underline{f} \in \mathcal{L}^1(\mathbb{R}^d, \mathbb{C}^k) \cap \mathcal{L}^2(\mathbb{R}^d, \mathbb{C}^k)$, the Fourier transform $\widehat{\underline{f}}$ of $\underline{f}$ is defined as

$$\widehat{\underline{f}}(\underline{\xi}) = \left(\frac{1}{2\pi}\right)^{d/2} \int_{\mathbb{R}^d} e^{-\iota \underline{\xi}^T \underline{x}} \underline{f}(\underline{x}) d\underline{x}, \qquad \underline{\xi} \in \mathbb{R}^d, \tag{1}$$

and the inverse Fourier transform $\check{\underline{f}}$ of $\underline{f}$ is

$$\check{\underline{f}}(\underline{x}) = \left(\frac{1}{2\pi}\right)^{d/2} \int_{\mathbb{R}^d} e^{\iota \underline{\xi}^T \underline{x}} \underline{f}(\underline{\xi}) d\underline{\xi}, \qquad \underline{x} \in \mathbb{R}^d. \tag{2}$$

We consider an initial boundary value problem for the Burgers equation on the plane:

$$\frac{\partial}{\partial t}\underline{u}(\underline{x}, t) - \nu \Delta \underline{u}(\underline{x}, t) + \left(\underline{u}^T(\underline{x}, t)\nabla\right)\underline{u}(\underline{x}, t) = \underline{f}(\underline{x}, t), \ \ \underline{x} \in \mathbb{R}^2, \ \ t > 0, \tag{3}$$

$$\underline{u}(\underline{x}, 0) = \underline{u}_0(\underline{x}), \qquad \underline{x} \in \mathbb{R}^2, \tag{4}$$

$$\lim_{|\underline{x}| \to \infty} \underline{u}(\underline{x}, t) = \underline{0}, \qquad t > 0, \tag{5}$$

where ∇ is the gradient operator, Δ is the Laplace operator, $\nu > 0$ is the viscosity, $\underline{u}_0 : \mathbb{R}^2 \to \mathbb{C}^2$ is the known initial solution, $\underline{f} = (f_1, f_2)^T$ is the source term, and $\underline{u} = (u_1, u_2)^T : \mathbb{R}^2 \times \mathbb{R}^+ \to \mathbb{C}^2$ is the unknown function.

We suppose that, for each $t > 0$, and for $s = 1, 2$,

$$\underline{u}(\cdot, t), \ \ \frac{\partial \underline{u}}{\partial t}(\cdot, t), \ \ \frac{\partial \underline{u}}{\partial x_s}(\cdot, t), \ \ u_s \frac{\partial \underline{u}}{\partial x_s}(\cdot, t) \in \mathcal{L}^1(\mathbb{R}^2, \mathbb{C}^2) \cap \mathcal{L}^2(\mathbb{R}^2, \mathbb{C}^2), \tag{6}$$

$$\lim_{|\underline{x}| \to \infty} \frac{\partial \underline{u}}{\partial x_s}(\underline{x}, t) = \underline{0}. \tag{7}$$

Let $\widehat{\underline{u}}(\underline{\xi}, t)$, $\underline{\xi} \in \mathbb{R}^2$, $t > 0$ be the Fourier transform of $\underline{u}(\underline{x}, t), \underline{x} \in \mathbb{R}^2, t > 0$, with respect to variable $\underline{x}$. Then, when $s = 1, 2$, $\underline{\xi} \in \mathbb{R}^2$, $t > 0$, we have

$$\widehat{\frac{\partial \underline{u}}{\partial t}}(\underline{\xi}, t) = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{-\iota \underline{\xi}^T \underline{x}} \frac{\partial}{\partial t} \underline{u}(\underline{x}, t) d\underline{x} = \frac{\partial}{\partial t} \widehat{\underline{u}}(\underline{\xi}, t), \tag{8}$$