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Unstable dimension variability structure in the parameter space of coupled Hénon maps

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ABSTRACT

Coupled maps have been investigated to model the applications of periodic and chaotic dynamics of spatially extended systems. We have studied the parameter space of coupled Hénon maps and showed that attractors possessing unstable dimension variability (UDV) appear for parameters neighbouring the so called shrimp domains, representing parameters leading to stable periodic behaviour. Therefore, the UDV should be very likely to be found in the same large class of natural and man-made systems that present shrimp domains.

1. Introduction

In the early 1960s, Lorenz proposed three coupled nonlinear differential equations to model atmospheric convection [1]. The Lorenz equations have attracted great attention due to their interesting dynamical solutions, for instance, a chaotic attractor [2,3]. Inspired by the Lorenz attractor, Hénon introduced a map from the Poincaré section of the Lorenz equations [4], the Hénon map, which despite being simpler than differential equations, still preserve most of the relevant features of the solutions of the Lorenz system. Hénon's map behaviour can be set by its parameters. A global two dimensional visualisation of the map's behaviour as a function of two parameters is called the parameter space. It indicates by colours the type of solutions found in the system for those two parameters. Same colour regions represent an open set of parameters that lead the system to the same behaviour. These open sets form domains in the parameter space, and appear organised in a very regular way. Gallas, studying the Hénon and another unidimensional map, Gallas [5,6] reported the existence of shrimps-shaped domains that appear along special directions and organised in a self-similar way representing parameters for which the maps present periodic stable behaviour. Many other works have reported shrimp-shaped domains in parameter spaces of other mathematical models, for instance, Hoff and collaborators [7] verified the isoperiodic structures in a four-dimensional Chua model with smooth nonlinearity. Periodic windows were observed in mathematical models of two-gene for gene expression and regulation [8]. It was also verified the existence of shrimp-shaped domains in an experimental

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Chua's circuit [9]. Medeiros and collaborators showed that periodic domains arose in the parameter space of an impact oscillator [10].

In this work, we study coupled Hénon maps. Coupled maps networks are one of the most investigated types of spatially extended systems where both space and time are discrete variables while the state variable is continuous. The system is formed by a local dynamical unit interacting with other units by means of a coupling architecture. Coupled maps networks have been considered in mathematical models to study biological neuronal networks [11–13], image encryption [14], secure communication [15], and spatiotemporal chaos [16].

Here, we aim at finding parameter regions responsible to make a coupled network of Hénon maps to present unstable dimension variability (UDV). The UDV describes a situation in which the dimension of the unstable and stable tangent spaces are not constant over the chaotic set [17,18]. Unstable dimension variability was introduced by Abraham and Smale by means of a two-dimensional continuously differentiable map [19]. If an attractor contains periodic orbits with different number of unstable directions, the attractor has UDV [18]. In 1997, Kostelich and collaborators [17] analysed the torus map, a simple two-dimensional map, that displays UDV. They showed analysis and simulations to illustrate the dynamics behaviours of systems with UDV. Noninvertible two-dimensional map can have an invariant set exhibiting UDV [18]. UDV in a three-dimensional map was observed by Dawson [20]. In this work, we consider coupled Hénon maps to study higher-dimensional dynamical systems. We build a network where the maps are globally coupled, known as all-to-all coupling. It is still allusive how periodic windows appear in coupled systems as the dimension of the system increases. This paper however shows that despite the system studied has dimension 6, these windows are still present appearing nearby regions with attractors possessing UDV.

Many mathematical models exhibit UDV phenomenon such as the double rotor [21], coupled chaotic maps [22], and periodically forced drift waves [23]. In the UDV, there is the absence of shadowing trajectories, resulting in that a long numerical trajectory may not be associated with any mathematically true solution [24,25]. This way, it is important to investigate UDV in chaotic systems to know if the computed solution is close to a true solution, namely if there is UDV, simulations will produce chaotic pseudo-solutions. Due to the fact that the trajectory moves in regions with different unstable dimensions, the values of the Lyapunov exponents oscillate between negative and positive values [26]. Dawson and collaborators [27] identified UDV in computer simulations of a double rotor. They verified that the second largest exponent fluctuates about zero. Therefore, we use the finite time Lyapunov exponents as a diagnostic of UDV [28].

This article is organised as follows: in Section 2 we present the model of coupled Hénon maps. In Section 3 we study domains of unstable dimension variability in the parameter space. In the last section, we draw the conclusions.

2. The mathematical model

The Hénon map is a two-dimensional discrete time dynamical system given by

$$\begin{aligned} x_{n+1} &= a - x_n^2 + b y_n, \\ y_{n+1} &= x_n, \end{aligned} \tag{1}$$

where x_n and y_n are the continuous states at discrete time-n ($n = 0, 1, 2, \dots$), a and b are positive parameters of the system. For the Hénon map over a range of a and b values it is possible to obtain chaotic, intermittent, or periodic behaviour. The chaotic behaviour can be identified when the largest Lyapunov exponent is positive [29]. The kth Lyapunov exponent is defined as

$$\lambda_k(\mathbf{x}_0, n) = \frac{1}{n} \ln(||\mathbf{Df}^n(\mathbf{x}_0)\mathbf{u_k}||), \tag{2}$$

where $\mathbf{Df}^n$ is the Jacobian matrix of the n-time iterated map evaluated at point $\mathbf{x}_0$ and $\mathbf{u_k}$ is the eigenvector corresponding to the kth eigenvalue of the Jacobian matrix. For infinite-times, we have the usual Lyapunov exponent

$$\lambda_k = \lim_{n\to\infty} \lambda_k(\mathbf{x}_0, n), \tag{3}$$

where the value is the same for almost every initial condition $\mathbf{x}_0$. To identify UDV, we consider the finite time-n Lyapunov exponent, calculated by making n in Eq. (2) small and finite.

Fig. 1(a) shows the parameter space for the Hénon map with the largest Lyapunov exponent represented by the colour scale. We can see periodic domains in black and grey scale regions immersed in chaotic regions ($\lambda_1 > 0$), and in the white region the trajectories diverge to infinity. The periodic structures, known as shrimp-shaped, can also be identified by the Kaplan–Yorke dimension (Fig. 1(b)) that is defined as

$$D = \begin{cases} 0 & \text{if there is no such } m, \\ m + \frac{1}{|\lambda_{m+1}|} \sum_{k=1}^{m} \lambda_k & \text{if } m < N, \\ N & \text{if } m = N, \end{cases} \tag{4}$$

where m is the greatest integer for which $\sum_{k=1}^{m} \lambda_k \geq 0$ and N is the dimension of the system.