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Unified convergence domains of Newton-like methods for solving operator equations

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ABSTRACT

We present a unified semilocal convergence analysis in order to approximate a locally unique zero of an operator equation in a Banach space setting. Using our new idea of restricted convergence domains we generate smaller Lipschitz constants than in earlier studies leading to the following advantages: weaker sufficient convergence criteria, tighter error estimates on the distances involved and an at least as precise information on the location of the zero. Hence, the applicability of these methods is extended. These advantages are obtained under the same cost on the parameters involved. Numerical examples where the old sufficient convergence criteria cannot apply to solve equations but the new criteria can apply are also provided in this study.

1. Introduction

Let $\mathcal{B}_1, \mathcal{B}_2$ be Banach spaces and let $\mathcal{D}$ be an open and convex subset of $\mathcal{B}_1$. Let $U(v, \rho), \bar{U}(v, \rho)$ stand for the open and closed balls in $\mathcal{B}_1$ with center $v \in \mathcal{B}_1$ and of radius $\rho > 0$.

In this study, we are concerned with the problem of locating a locally unique zero x^* of the operator $F : \mathcal{D} \longrightarrow \mathcal{B}_2$. This problem is very important, since many problems can be formulated to find such a zero x^* using Mathematical Modeling [8,10–12,14,18,20]. It is well known that the zero x^* can be found in explicit form only in special cases. That is why most solution methods are usually iterative.

Newton's method defined for each $n = 0, 1, 2, \cdots$ by

$$x_{n+1} = x_n - F'(x_n)^{-1}F(x_n) \tag{1.1}$$

is undoubtedly the most popular iterative method for generating a sequence $\{x_n\}$ approximating x^* [1,2,4–20].

The most famous semilocal convergence results for Newton's method are the celebrated Newton–Kantorovich theorem and its refinement [8,14,20].

Theorem 1.1 [14]. *Let $F : \mathcal{D} \subset \mathcal{B}_1 \longrightarrow \mathcal{B}_2$ be a Fréchet-differentiable operator. Suppose that there exist $x_0 \in \mathcal{D}, K > 0, \beta \geq 0$ such that for each $x \in \mathcal{D}, F'(x_0)^{-1} \in L(\mathcal{B}_2, \mathcal{B}_1)$*

$$\|F'(x_0)^{-1}F(x_0)\| \leq \beta, \tag{1.2}$$

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$$\|F'(x_0)^{-1}(F'(x)-F'(y))\| \le K\|x-y\| \text{ for each } x\in\mathcal{D} \tag{1.3}$$

$$H=\beta K \le \frac{1}{2} \tag{1.4}$$

and

$$\bar{U}(x_0,\bar{r})\subseteq\mathcal{D}, \tag{1.5}$$

where

$$\bar{r}=\frac{1-\sqrt{1-2H}}{K}. \tag{1.6}$$

Then, the sequence $\{x_n\}$ generated by Newton's method is well defined in $\bar{U}(x_0,\bar{r})$, remains in $\bar{U}(x_0,\bar{r})$ for each $n=0,1,2,\cdots$ and converges to a unique zero $x^*\in\bar{U}(x_0,\bar{r})$ of operator F. Moreover, the following estimates hold:

$$\|x_{n+1}-x_n\|\le v_{n+1}-v_n$$

and

$$\|x_n-x^*\|\le v^*-v_n,$$

where sequence $\{v_n\}$ is defined by

$$v_0=0, v_1=\beta, v_{n+2}=v_{n+1}-\frac{K(v_{n+1}-v_n)^2}{2(1-Kv_{n+1})}=v_{n+1}-\frac{\psi(t_{n+1})}{\psi'(t_{n+1})}$$

and function ψ is defined by

$$\psi(t)=\frac{K}{2}t^2-t+\beta$$

and satisfies $\psi(\bar{r})=0$.

Numerous authors, since then have presented similar semilocal convergence results (see [11–14,20]). We also have weakened the sufficient convergence criterion and improved on the error estimates and the location of the zero x^* [3–9]. In the present study we are motivated by the works in [11,15,16,19] on the semilocal convergence of Newton-like methods. It is well known that the convergence domain of these methods is small in general. We introduce more precise convergence domains where the iterates lie than in [11,15,16,19] leading to smaller Lipschitz constants which yield to the advantages to our new approach as already stated in the introduction of this study.

The rest of the paper is organized as follows: Section 2 contains the semilocal convergence analysis of Newton-like methods, whereas the numerical examples and special cases are presented in the concluding Section 3.

2. Semi-local convergence: Fréchet-differentiable operators

Let $\beta\ge 0$, $K_0>0$, $K>0$. We present the improvement of Theorem 1.1. In view of (1.3) there exists $K_0>0$ such that for each $x\in D$

$$\|F'(x_0)^{-1}(F'(x)-F'(x_0))\|\le K_0\|x-x_0\|. \tag{2.1}$$

Notice that

$$K_0\le K \tag{2.2}$$

holds in general and $\frac{K}{K_0}$ can be arbitrarily large [4]. In a series of papers [4–9] we showed that (1.4) can be improved:

Theorem 2.1 [7]**.** *Let $F:\mathcal{D}\subset\mathcal{B}_1\longrightarrow\mathcal{B}_2$ be a Fréchet-differentiable operator. Suppose that there exist $x_0\in\mathcal{D}, K_0>0, K>0$ such that (1.2), (1.3), (2.1)*

$$H_0=\frac{1}{8}(4K_0+\sqrt{K_0K+8K_0^2}+\sqrt{K_0K})\beta\le\frac{1}{2} \tag{2.3}$$

and

$$\bar{U}(x_0,\bar{r}_0)\subseteq\mathcal{D} \tag{2.4}$$

hold. Then, the sequence $\{x_n\}$ generated by Newton's method is well defined in $\bar{U}(x_0,\bar{r}_0)$, remains in $\bar{U}(x_0,\bar{r}_0)$ for each $n=0,1,2,\cdots$ and converges to a unique zero $x^*\in\bar{U}(x_0,\bar{r}_0)$ of operator F. Moreover, the following estimates hold:

$$\|x_{n+1}-x_n\|\le u_{n+1}-u_n \tag{2.5}$$

and

$$\|x_n-x^*\|\le u^*-u_n, \tag{2.6}$$