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Odd components of co-trees and graph embeddings

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ABSTRACT

In this paper we investigate the relation between odd components of co-trees and graph embeddings. We show that any graph G must share one of the following two conditions: (a) for each integer h such that G may be embedded on S_h, the sphere with h handles, there is a spanning tree T in G such that $h = \frac{1}{2}(\beta(G) - \omega(T))$, where $\beta(G)$ and $\omega(T)$ are, respectively, the *Betti number* of G and the number of components of $G - E(T)$ having odd number of edges; (b) for every spanning tree T of G, there is an orientable embedding of G with exact $\omega(T) + 1$ faces. This extends Xuong and Liu's theorem [9,13] to some other (possible) genera. Infinitely many examples show that there are graphs which satisfy (a) but (b). Those make a correction of a result of Archdeacon [2, Theorem 1].

1. Introduction

Graphs here are simple connected. All the terms and notations are standard and may be found in [3].

A *surface* is a compact 2-manifold. An *orientable surface*, denoted by S_g, is the sphere with g handles. An *embedding* of a graph G is a drawing of G in a surface Σ such that no edge-crossing is permitted and each component of $\Sigma - G$ is an open disc. Let T be a spanning tree of G. By $\omega(T)$ we mean the number of the components of the co-tree $G - E(T)$ having odd number of edges. It is well known that odd components of co-trees play a key role in graph embeddings and there have been many literatures for it. The following result is due to Xuong and Liu [9,13].

Theorem A. *Let G be a graph. Then the maximum genus of G is*

$$\gamma_M(G) = \frac{1}{2}(\beta(G) - \min\{\omega(T)\}),$$

where $\beta(G)$ is the Betti number of G and the min is taken over all the spanning trees in G.

Another well known result is the following *Duke's interpolation theorem* [5] for genera of orientable surfaces on which a graph may be embedded.

Theorem B. *If a graph G may be embedded in S_h and $S_k (h \le k)$, then it also may be embedded in S_g for $g = h, h+1, \ldots, k$.*

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Theorem B says that the genera of orientable surfaces on which a graph may be embedded form a series of consecutive integer numbers. We write $[\gamma(G), \gamma_M(G)]$ as the *genera interval* of a graph G, where $\gamma(G)$ and $\gamma_M(G)$ are, respectively, the minimum genus (or *genus* in short) and the maximum genus of G.

One of the main results of this paper is to establish an interpolation theorem for odd components of co-trees in a graph, i.e., the following

Theorem 1. *Let G be a graph with two spanning trees T_1 and T_2. Then we have that*

(a) $\omega(T_1) \equiv \omega(T_2)(mod 2)$ *and*
(b) *for each integer m with $\omega(T_1) \le m \le \omega(T_2)$ and $m \equiv \omega(T_1)$, there is a spanning tree T in G such that $\omega(T) = m$.*

If we denote $g = \frac{1}{2}(\beta(G) - \omega(T))$ for each spanning tree T in G, then Theorem 1 shows that all the integers defined this way form a collection of consecutive integers. Define $[g_m(G), g_M(G)]$ as the *odd components interval* of G, where

$$g_m(G) = \frac{1}{2}(\beta(G) - max\{\omega(T)\}),$$

$$g_M(G) = \frac{1}{2}(\beta(G) - min\{\omega(T)\}).$$

It follows from Theorem A that $g_M = \gamma_M(G)$. Thus, we have that for each graph G, either $[\gamma(G), \gamma_M(G)] \subseteq [g_m(G), g_M(G)]$ or $[g_m(G), g_M(G)] \subseteq [\gamma(G), \gamma_M(G)]$. If $[\gamma(G), \gamma_M(G)] \subseteq [g_m(G), g_M(G)]$, then for every surface S_h with $\gamma(G) \le h \le \gamma_M(G)$, there is a spanning tree T such that

$$h = \frac{1}{2}(\beta(G) - w(T)).$$

Otherwise, for each spanning tree T in G, there is an orientable surface S_h on which G may be embedded and have exact $\omega(T) + 1$ faces, i.e., the following

Theorem 2. *Let G be a connected graph with the parameters $\gamma(G)$, $\gamma_M(G)$, g_m and g_M defined as above. Then G must satisfy one of the following conditions:*

(a) *for every surface S_h on which G may be embedded, there is a spanning tree T in G such that $h = \frac{1}{2}(\beta(G) - \omega(T))$;*
(b) *for every spanning tree T of G, G may be embedded in some orientable surface which has exact $\omega(T) + 1$ faces.*

Remark (1)**.** Theorem 2 does extend Xuong and Liu's result to some other (possible) surfaces since for each integer $h \in [\gamma(G), \gamma_M(G)] \bigcap [g_m(G), g_M(G)]$, there is a spanning tree T such that $h = \frac{1}{2}(\beta(G) - \omega(T))$; (2) the readers with care may notice that the partial result of Theorem 2 (i.e., (b) of Theorem 2) was stated in Archdeacon's paper [2]. Here, what we will see is that there are infinitely many graphs which do not satisfy the condition (b) of Theorem 2. Hence, Theorem 2 also makes a correction of a result in [[2], Theorem 1]. (3) Theorem 2 may find its uses in graphs with a spanning tree whose co-tree has large number of odd components. In particular, we have the following result:

Corollary 3. *Let G be a cubic hamiltonian graph. If G is nonplanar, then for every orientable surface S_g on which G may be embedded, there is a spanning tree T in G such that $g = \frac{1}{2}(\beta(G) - \omega(T))$.*

2. Proof of main results

In this section we shall prove Theorems 1 and 2. We first present the following result for tree-transformation and their odd component numbers of co-trees.

Lemma 1. *Let G be a connected graph with a spanning tree T. Let e be an edge in $G - E(T)$ and f is another edge in $C_e - e$, where C_e is the unique cycle in $T + e$. Then $T' = T + e - f$ is another spanning tree in G such that $\omega(T') \equiv \omega(T)(mod\, 2)$ and $|\omega(T') - \omega(T)| \le 2$.*

Proof. Let σ_e be the component of $G - E(T)$ containing e. Let σ_x and σ_y be, respectively, the (possible) two components of $G - E(T)$ containing x and y, where $f = (x, y)$. Then there are several more cases should be handled as listed below.

(1) σ_e, σ_x and σ_y are pairwisely distinct components;
(2) $\sigma_x = \sigma_y \neq \sigma_e$;
(3) $\sigma_x = \sigma_e \neq \sigma_y$;
(4) $\sigma_y = \sigma_e \neq \sigma_x$;
(5) $\sigma_y = \sigma_x = \sigma_e$.

Now we consider the case that σ_e, σ_x and σ_y are pairwisely distinct components. We concentrate on the case (1). There are two subcases to be handled. □

Subcase 1.1. $\sigma_e - e$ is disconnected.
Let $\sigma_e - e = \sigma_e' - \sigma_e''$.