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Uniform convergence and order reduction of the fractional implicit Euler method to solve singularly perturbed 2D reaction-diffusion problems

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A B S T R A C T

In this paper we analyze the uniform convergence of a numerical method designed to approximate efficiently the solution of 2D parabolic singularly perturbed problems of reaction diffusion type. The method combines a modified fractional implicit Euler method to discretize in time, and the classical central finite difference scheme, on a special nonuniform mesh, to discretize in space. The resulting fully discrete scheme is uniformly convergent with respect to the diffusion parameter. The analysis of the convergence is made by using a two step technique, which discretizes first in time and later on in space. We show the order reduction phenomenon associated to the fractional implicit Euler method, which typically appears if the boundary conditions are time dependent and a natural evaluation of them is done. An appropriate choice for the boundary conditions is proposed and analyzed in detail, proving that the order reduction can be removed. Some numerical tests show the practical effects of our method; as well, we compare it with the classical choice for the boundary data in terms of the uniform consistency and the order of uniform convergence of the numerical scheme.

1. Introduction

We consider 2D reaction-diffusion parabolic problems modeled by the differential equation

$$\frac{\partial u}{\partial t} - \varepsilon^2 \Delta u + ku = f(x, y, t),$$

where $0 < \varepsilon \leq 1$ and the reaction term $k \equiv k(x, y, t)$, is a smooth function such that $k \geq \beta^2 > 0, \beta > 0$. We will focus our attention on the case $\varepsilon \ll \beta$, for which, in general, their solutions have a multiscale character, changing rapidly in certain narrow regions called boundary layers (see [13,17]). If standard finite difference or finite element methods are used on uniform meshes, for sufficiently small values of ε the numerical approximations are coarse. So, uniformly convergent methods, for which the rate of convergence and the constant's error of the method are both independent of ε, are convenient to obtain reliable numerical approximations for any value of the diffusion parameter.

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Many numerical uniformly convergent methods are developed in last years, by using fitted operator methods (see [14,17] and references therein) or fitted mesh methods (see [9,13,19,20] and references therein) for different types of singularly perturbed problems. The case of 2D elliptic problems was analyzed, for instance, in [4,11,12]; for 2D time dependent problems, in [2,3,6,7] it was described a two step technique to construct the fully discrete method, which discretizes firstly in time and later on in space. Then, using the direction alternating technique (see [21]), only tridiagonal linear systems must be solved to obtain the numerical solution. Nevertheless, in most of papers, only homogeneous boundary conditions were considered, to avoid the order reduction phenomenon, which is specially severe if natural evaluations of the boundary conditions are used.

Recently, in [5], the two step technique in reverse order, i.e., firstly in space and secondly in time, was considered, proving that the uniform convergence is reached when the discretization in space uses classical finite difference schemes defined on an appropriate piecewise uniform Shishkin mesh, together with the classical backward Euler method, defined on a uniform mesh, to discretize in time. In that case, no order reductions occur, but the computational cost of that scheme is substantially large because large pentadiagonal linear systems, one per time step, must be solved. To reduce that computational cost, in this paper we consider a method of type alternating directions. So, the fractional implicit Euler method is used to discretize in time, in combination with a splitting of the diffusion-reaction operator which separates the derivatives of the spatial variables. The analysis of the uniform convergence of the fully discrete scheme follows similar ideas to those ones in [6]; the main differences appear in the proof of the uniform consistency of the time semidiscretization stage using the fractional implicit Euler method. It is well known (see [1] and references therein) that a classical evaluation of the boundary conditions, when most of the one step methods are used to integrate in time, causes a reduction in the order of convergence. This order reduction is specially severe when the fractional implicit Euler method is used in problems where non homogeneous time dependent boundary conditions appear. So, here we propose a suitable modification of these evaluations which eliminates the order reduction.

The paper is structured as follows. In Section 2, we set up the problem to be solved and we remind some results concerning to the asymptotic behavior, with respect to ε, of the exact solution and its derivatives. In Section 3, we introduce the time semidiscretization, which uses the fractional implicit Euler method, and we prove its uniform convergence. When non homogenous boundary conditions appear, suitable evaluations of the boundary data are essential to avoid the order reduction. As well, we study the asymptotic behavior of the exact solution of the semidiscrete problems resulting after the time discretization, which preserve, in essence, the same ε-asymptotic behavior that the solution of the continuous problem. In Section 4, we define the spatial discretization by using a finite differences scheme on a piecewise uniform mesh of Shishkin type, and we remind the techniques used in previous studies for proving its uniform convergence. Combining the results of the two stages of discretization, we deduce the uniform convergence of the fully discrete scheme. Finally, in Section 5, some numerical results are shown, which corroborate in practice the efficiency of the method, the order of uniform of convergence and the influence of the order reduction phenomenon.

Henceforth, C denotes a generic positive constant independent of the diffusion parameter ε and also of the discretization parameters N and M. We always use the pointwise maximum norm, denoted by $\|\cdot\|_D$ (where D is the corresponding domain).

2. Asymptotic behavior of the continuous problem

Let $\Omega \equiv (0, 1) \times (0, 1)$. We consider the initial-boundary value problem

$$\begin{aligned} &\mathcal{L}u \equiv \frac{\partial u}{\partial t} + \mathcal{L}_{1,\varepsilon}u + \mathcal{L}_{2,\varepsilon}u = f_1(x,y,t) + f_2(x,y,t), \text{ in } \Omega \times (0,T], \\ &u(x,y,0) = \varphi(x,y), \text{ in } \Omega, \\ &u(x,y,t) = g(x,y,t), \text{ in } \partial\Omega \times [0,T], \end{aligned} \tag{1}$$

where the spatial differential operators $\mathcal{L}_{1,\varepsilon}$, $\mathcal{L}_{2,\varepsilon}$ are defined by

$$\mathcal{L}_{1,\varepsilon}u \equiv -\varepsilon^2 \frac{\partial^2 u}{\partial x^2} + k_1(x,y,t)u, \ \mathcal{L}_{2,\varepsilon}u \equiv -\varepsilon^2 \frac{\partial^2 u}{\partial y^2} + k_2(x,y,t)u, \tag{2}$$

with $f_1 + f_2 = f,\ k = k_1 + k_2,\ k_i \ge \beta_i^2,\ \beta_i > 0,\ i = 1, 2.$

We assume that functions $k_i,\ f_i, i = 1, 2,\ g$ and φ are sufficiently smooth and also that sufficient compatibility conditions hold among them in order to $u(x, y, t) \in C^{4,2}(\Omega \times (0, T])$, i.e., it has continuous derivatives up to fourth order in space and second order in time (see [5–7]). It is well known that the exact solution of (1), in general, has boundary layers at the four sides of the spatial domain. Moreover, the solution can be decomposed in the form

$$u = u_0 + \sum_{p=1}^{4} u_p + \sum_{p=1}^{4} w_p,$$

where u_0 is the regular component, $u_p,\ p = 1, 2, 3, 4,$ are the edge boundary layer functions associated at each one of the four sides of the unit square and $w_p,\ p = 1, 2, 3, 4,$ are the corner layer functions corresponding to the corner points