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On piecewise continuous solutions of higher order impulsive fractional differential equations and applications

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ABSTRACT

For impulsive differential equations with fractional order, we show that the formula of solutions in cited papers are incorrect. We then give exact piecewise continuous solutions (the explicit solutions) of two classes of fractional differential equations of order $\alpha \in (n-1, n)$ involving Caputo derivatives and Riemann–Liouville derivatives. Apply our results to obtain existence of solutions of two classes of initial value problems of singular impulsive fractional differential equations. Examples are presented to illustrate the main theorems.

1. Introduction

Recently, in [5], Feckan and Zhou pointed out that the formula of solutions for impulsive fractional differential equations in [1–4] is incorrect and gave their correct formula. In [11,13], the authors established a general framework to find the solutions for impulsive fractional boundary value problems and obtained some sufficient conditions for the existence of the solutions to a kind of impulsive fractional differential equations respectively. In [12], the authors illustrated their comprehensions for the counterexample in [5] and criticized the viewpoint in [5,11,13]. Next, in [7], Feckan et al. expounded for the counterexample in [5] and provided further five explanations in the paper.

In [14], Zhang et al. tried to find out the exact formula of the general solution for impulsive Cauchy problem with Caputo fractional derivative $q \in (0, 1)$ of the form

$$\begin{cases} {}^CD^q_{0^+}x(t) = f(t, x(t)), t \in [0, T], t \neq t_k, k = 1, 2, \ldots, m, \\ \Delta x|_{t=t_k} = I_k(x(t_k^-)), k = 1, 2, \ldots, m, \\ x(0) = x_0, \end{cases} \tag{1}$$

where ${}^CD^q_{0^+}$ is the standard Caputo fractional derivative, f: $[0, T] \times R \to R$ is appropriate continuous function to be specified later, $I_k : R \to R (k = 1, 2, \ldots, m)$ are appropriate functions, and $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta x|_{t=t_k} = \lim_{t\to t_k^+} x(t) - \lim_{t\to t_k^-} x(t) = x(t_k^+) - x(t_k^-)$ and $x(t_k^+), x(t_k^-)$ represent the right and left limits of $x(t)$ at $t = t_k$ respectively, x_0 a constant.

We say that a function x: $(0, T] \to R$ is a solution of system (1) if x is continuous on $(t_i, t_{i+1}) (i = 0, 1, 2, \ldots, m)$, $\lim_{t\to 0^+} x(t) = x(0) = x_0$, $\lim_{t\to t_k^+} x(t) - \lim_{t\to t_k^-} x(t) = I_k(x(t_k^-)), k = 1, 2, \ldots, m$, and ${}^CD^q_{0^+}x(t) = f(t, x(t))$ on $[0, T]$. The main result in [14] is as follows:

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Theorem Z (Theorem 2.1 in [14]). *Let $q \in (0, 1)$ and h be a constant. Then system (1) is equivalent with integral equation*

$$x(t) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds, \ t \in [0, t_1], \\ x_0 + \sum\limits_{k=1}^{n} I_k(x(t_k^-)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds \\ \quad + \frac{h}{\Gamma(q)} \sum\limits_{=1}^{n} \Big[I_k(x(t_k^-)) \Big(\int_0^{t_k} (t_k - s)^{q-1} f(s, x(s)) ds \\ \quad + \int_{t_k}^t (t-s)^{q-1} f(s, x(s)) ds - \int_0^t (t-s)^{q-1} f(s, x(s)) ds \Big) \Big], \\ t \in (t_n, t_{n+1}], n = 1, 2, \ldots, m \end{cases} \tag{2}$$

provided the integrals in (2) exist.

We find that this theorem is wrong too. We will prove by using mathematical induction method that x is a solution of system (1) if and only if x satisfies the following integral equation

$$x(t) = \begin{cases} x_0 + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s, x(s)) ds, \ t \in [0, t_1], \\ \sum\limits_{k=1}^{n} I_k(x(t_k)) + x_0 + \int_0^t (t-s)^{q-1} f(s, x(s)) ds, t \in (t_n, t_{n+1}], n = 1, 2, \ldots, m \end{cases} \tag{3}$$

under the following weak assumption (see Claim L in Appendix section at the end of the paper). It is easy to see that (2) is not correct. This mistake comes from the transformation from initial value problem into integral equation. So it is interesting to establish existence result for IVP(1) by a new method.

Motivated by [14], we firstly consider the exact piecewise continuous solutions of the following fractional differential equations

$$^{c}D_{0^+}^{\alpha} x(t) = f(t), t \in (t_i, t_{i+1}], i \in N_0, \tag{4}$$

and

$$D_{0^+}^{\alpha} x(t) = g(t), t \in (t_i, t_{i+1}], i \in N_0, \tag{5}$$

where $\alpha \in (n-1, n)$, $^{c}D_{0^+}^{*}$ denotes the standard Caputo fractional derivative of order $*$ with the start points $t = 0$ and $D_{0^+}^{*}$ the standard Riemann-Liouville fractional derivative of order $*$ with the start points $t = 0$, $b > 0$ and $0 = t_0 < t_1 < t_2 < \cdots < t_m < t_{m+1} = b$, $N_0 = \{0, 1, 2, \ldots, m\}$, f, g: $(0, 1) \to R$ are continuous and satisfy

(i) $|f(t)| \le t^k (b-t)^l$ with $k > -1$, $l \le 0$ with $l > \max\{-\alpha, -\alpha - k\}$;
(ii) $|g(t)| \le t^k (b-t)^l$ with $k > -1$ and $l \le 0$ with $l > \max\{-\alpha, -n - k\}$.

A function x: $(0, b] \to R$ is called a piecewise continuous solution of a fractional differential Eq. (4) (or (5)) if $x|_{(t_i, t_{i+1}]} (i \in N_0)$ is continuous, $\lim_{t \to t_i^+} x(t) (i \in N_0)$ (or $\lim_{t \to t_i^+} (t - t_i)^{1-\alpha} x(t) (i \in N_0)$) exist and x satisfies all equations in (4) (or (5)). The first purpose of this paper is to given the explicit solution (piecewise continuous solution) of (4) and (5) respectively. The second purpose of this paper is to apply our results to establish new existence results for the following two impulsive initial value problems

$$\begin{cases} ^{C}D_{0^+}^{q} x(t) = F(t, x(t)), t \in (t_i, t_{i+1}], i \in N_0, \\ \Delta x|_{t=t_i} = I(t_i, x(t_i)), i \in N, \\ x(0) = x_0, \end{cases} \tag{6}$$

and

$$\begin{cases} D_{0^+}^{q} x(t) = G(t, x(t)), t \in (t_k, t_{k+1}], k \in N_0, \\ \lim\limits_{t \to t_i^+} (t - t_i)^{1-\alpha} x(t) = J(t_i, x(t_i)), i \in N, \\ \lim\limits_{t \to 0^+} t^{1-q} x(t) = x_0, \end{cases} \tag{7}$$

where $q \in (0, 1)$, $^{C}D_{0^+}^{q}$ is the standard Caputo fractional derivative, $D_{0^+}^{q}$ is the standard Riemann-Liouville fractional derivative, F, G: $(0, T) \times R \to R$ are strong Carathéodory functions (see Definitions 2.4 and 2.5), I, J: $\{t_k$: $k \in N\} \times R \to R$ are discrete Carathéodory functions (see Definitions 2.6 and 2.7), $T > 0$, $0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = T$, $\Delta x|_{t=t_k} = \lim_{t \to t_k^+} x(t) - x(t_k)$, x_0 a constant.

A function x: $(0, T] \to R$ is called a solution of IVP(6) (or IVP(7)) if $x|_{(t_i, t_{i+1}]} (i \in N_0)$ is continuous, $\lim_{t \to t_i^+} x(t) (i \in N_0)$ (or $\lim_{t \to t_i^+} (t - t_i)^{1-q} x(t) (i \in N_0)$) exist and x satisfies all equations in (6) (or (7)).

It must be noted that in [8] and other cited papers, the solution of $^{c}D_{0^+}^{\alpha} x(t) = f(t), t \in (0, b]$ (or $D_{0^+}^{\alpha} x(t) = g(t), t \in (0, b]$) is obtained under the assumption that $f \in L_1(0, b)$ (or $g \in L_1(0, b)$. However when we consider the equation $^{C}D_{0^+}^{\alpha} x(t) = t^{-\frac{1}{2}} (1-t)^{-\frac{3}{2}}, t \in (0, 1), \alpha \in (3, 4)$ with boundary conditions $x(0) = x'(0) = 0$, we can get its solutions (see [10]):

$$x(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} s^{-\frac{1}{2}} (1-s)^{-\frac{3}{2}} ds + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, c_1, c_2 \in R.$$