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Regularization and implicit Euler discretization of linear-quadratic optimal control problems with bang-bang solutions

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ABSTRACT

We analyze the implicit Euler discretization for a class of convex linear-quadratic optimal control problems with control appearing linearly. Constraints are defined by lower and upper bounds for the controls, and the cost functional may depend on a regularization parameter ν. Without any structural assumption on the optimal control we prove convergence of order 1 w.r.t. the mesh size for the discrete optimal values. Under the additional assumption that the optimal control is of bang-bang type and the switching function satisfies a growth condition around their zeros we show that the solutions are calm functions of perturbation and regularization parameters. By applying this result to the implicit Euler discretization we improve existing error estimates for discretizations based on the explicit Euler method. Numerical experiments confirm the theoretical findings and demonstrate the usefulness of implicit methods and regularization in case of bang-bang controls.

1. Introduction

Perturbation and discretization of optimal control problems are well studied for the case that the optimal control is sufficiently smooth (see e.g. Dontchev and Hager [1], Dontchev et al. [2], Dontchev and Rockafellar [3], Malanowski [4,5], Malanowski et al. [6], Alt [7] for control problems governed by ordinary differential equations and Tröltzsch [8,9] and the papers cited therein for control problems governed by partial differential equations). The results are usually based on second-order optimality conditions. Due to the lack of such conditions for bang-bang controls, there have been only a few papers on discretization of such controls (see Alt and Mackenroth [10], Dhamo and Tröltzsch [11] and the papers cited therein). New second-order optimality conditions for bang-bang controls have been developed recently in Felgenhauer [12–14], Maurer et al. [15], and Osmolovskii and Maurer [16–18] (see also the papers cited therein), and variants of these conditions have then been used in Veliov [19], Alt et al. [20], Alt and Seydenschwanz [21], and in Seydenschwanz [22] to obtain error estimates for discretization of optimal control problems governed by ordinary differential equations and in Deckelnick and Hinze [23] for elliptic control problems.

Discretization combined with regularization is a good alternative to direct discretization, since the problem to be solved is replaced by problems having smoother solutions. The regularization by an L^2-term in the cost functional of optimal

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control problems has been intensively studied during the last years (see e.g. Wachsmuth [24] and the papers cited therein). The dependency of solutions on regularization parameters and the combination with discretization has been investigated in Hager [25] for multiplier methods, in Alt and Seydenschwanz [26], Seydenschwanz [22] for Euler discretization of control problems governed by ordinary differential equations, and in Lorenz and Rösch [27] for elliptic control problems with state constraints. Results for control problems with a sparsity functional can be found e.g. in Alt and Schneider [28] for control problems governed by ordinary differential equations, and in Wachsmuth and Wachsmuth [29] for control problems governed by partial differential equations. A duality approach has been investigated in Alt et al. [30].

It is well-known that the integration of stiff ODEs by explicit Euler discretization gives rise to peculiar difficulties. Thus, explicit Euler may be not suitable as discretization scheme for optimal control problems governed by stiff ODEs (compare Alt and Seydenschwanz [21, Example 7.1], Seydenschwanz [31, Beispiel 3.2.17, Beispiel 4.2.14]). Among others, such problems arise in chemical reaction kinetics. For a further detailed discussion on this topic, we refer the reader to Stoer and Bulirsch [32, Section 7.2.16]. Surprisingly, only a few papers deriving error estimates for discretized control problems are concerned with implicit discretizations. Hager [33] and Dontchev et al. [34] investigate general Runge–Kutta methods assuming that the optimal control is Lipschitz continuous. For problems with bang-bang optimal controls Alt and Seydenschwanz [21] and Seydenschwanz [31] seem to be the only papers up to now dealing with an implicit discretization method based on the implicit midpoint rule. We will therefore study in the following a discretization scheme based on implicit Euler method, which is often used for practical computations.

The technique used here to derive error estimates for the discretized optimal controls allows to improve the error estimates of Seydenschwanz [22] for discretizations based on the explicit Euler method, and numerical examples demonstrate that the error estimates obtained here are sharp. As auxiliary results we derive estimates for optimal values and solutions of perturbed control problems.

We use the following notations: $\mathbb{R}^n$ is the n-dimensional Euclidean space with the inner product denoted by $\langle x, y\rangle$ and the norm $|x| = \langle x, x\rangle^{1/2}$. For an $m \times n$-matrix M we denote the spectral norm by $\|M\| = \sup_{|z|\le 1} |Mz|$. Let $t_0, t_f \in \mathbb{R}$, $t_0 < t_f$. We denote by $L^2(t_0, t_f; \mathbb{R}^m)$ the Hilbert space of square integrable, measurable vector functions $u : [t_0, t_f] \to \mathbb{R}^m$ with

$$\|u\|_2 = \left(\int_{t_0}^{t_f} |u(t)|^2 \, \mathrm{d}t\right)^{\frac{1}{2}} < \infty,$$

by $L^1(t_0, t_f; \mathbb{R}^m)$ the Banach space of integrable, measurable vector functions $u : [t_0, t_f] \to \mathbb{R}^m$ with

$$\|u\|_1 = \int_{t_0}^{t_f} \sum_{i=1}^{m} |u_i(t)| \, \mathrm{d}t = \sum_{i=1}^{m} \|u_i\|_1 < \infty,$$

and $L^\infty(t_0, t_f; \mathbb{R}^m)$ is the Banach space of essentially bounded vector functions with the norm

$$\|u\|_\infty = \max_{1\le i\le m} ess\,sup_{t\in[t_0,t_f]} |u_i(t)|.$$

For $p \in \{1, 2, \infty\}$ we denote by $W^1_p(t_0, t_f; \mathbb{R}^n)$ the spaces of absolutely continuous functions on $[t_0, t_f]$ with derivative in $L^p(t_0, t_f; \mathbb{R}^n)$, i.e.

$$W^1_p(t_0, t_f; \mathbb{R}^n) = \{x \in L^p(t_0, t_f; \mathbb{R}^n) \mid \dot{x} \in L^p(t_0, t_f; \mathbb{R}^n)\}$$

with

$$\|x\|_{1,p} = \left(|x(t_0)|^p + \|\dot{x}\|_p^p\right)^{\frac{1}{p}}$$

for $p = 1, 2$ and

$$\|x\|_{1,\infty} = \max\{\|x\|_\infty, \|\dot{x}\|_\infty\}.$$

Let $X = X_1 \times X_2$, where $X_1 = W^1_\infty(t_0, t_f; \mathbb{R}^n)$, $X_2 = L^\infty(t_0, t_f; \mathbb{R}^m)$, and

$$\|(x, u)\| = \max\{\|x\|_{1,\infty}, \|u\|_\infty\}.$$

We consider a class of convex linear-quadratic optimal control problems with control appearing linearly, where the constraints are described by lower and upper bounds for the controls. We want to solve the control problem numerically based on implicit Euler discretization combined with regularization, and we are interested in the dependence of the solutions of the discretized problems on the mesh size and the regularization parameter. As we shall see in Section 6 the discretized control problems may be interpreted as perturbations of the original problem. Therefore, we consider the following class of control problems depending on a perturbation parameter $p = (\xi, \xi_0, \xi_f, \zeta, \eta) \in X_3$, where

$$X_3 = L^1(t_0, t_f; \mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \times L^\infty(t_0, t_f; \mathbb{R}^m) \times L^\infty(t_0, t_f; \mathbb{R}^n)$$

and a regularization parameter $\nu \in X_4 = \{\nu \in \mathbb{R} \mid \nu \ge 0\}$: