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Stability analysis of a parametric family of iterative methods for solving nonlinear models[☆]

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A B S T R A C T

A one-parametric family of fourth-order iterative methods for solving nonlinear systems is presented, proving the fourth-order of convergence of all members in this family, except one of them whose order is five. The methods in our family are numerically compared with other known methods in terms of the classical efficiency index (order of convergence and number of functional evaluations) and in terms of the operational efficiency index, which also takes into account the total number of product-quotients per iteration. In order to analyze its stability and its dynamical properties, the parameter space for quadratic polynomials is shown. The stability of the strange fixed points is studied in this case. We note that even for this particular case, the family presents a very interesting dynamical behavior. The analysis of the parameter plane allows us to find values for the involved parameter with good stability properties as well as other values with bad numerical behavior. Finally, amongst the first ones, there is a special value of the parameter related to a fifth-order method in the family.

1. Introduction

The analysis of scalar nonlinear equations has a well-developed mathematical, numerical and computational theory. However, many of the applied problems in Science and Engineering are modeled by systems of nonlinear equations. This situation is more complicated than the scalar one and it has been less studied.

Let us consider the problem of finding a zero of a function $F : D \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, that is, a solution $\bar{x} \in D$ of the nonlinear system $F(x) = 0$, of n equations with n variables. This solution can be obtained as a fixed point of some function $\bar{G} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ by means of a fixed-point iteration method. Newton's scheme is the most known and used method of this type.

In the last decades, some researchers have proposed different iterative schemes as an alternative to Newton's method. These variants have been designed by means of different techniques, providing in the most of cases multistep schemes. We can mention, among others, the known Jarratt's method of order four [1] as well as the schemes published recently by F. Awawdeh [2], by J.R. Sharma et al. [3,4] of orders four and five, by V. Kanwar et al. [5] also of order four, etc. and the references therein.

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As the order of an iterative method increases, so does the number of functional evaluations per step. The efficiency index, introduced by Ostrowski in [6] gives a measure of the balance between these quantities, according to the formula $I = p^{1/d}$, where p is the order of convergence of the method and d the number of functional evaluations per step. However, the complexity of the iterative scheme used is a key factor when a nonlinear system of equations must be solved. So, not only the functional evaluations and the order of convergence must be taken into account, but also the amount of product-quotients per step. The resulting computational efficiency index was defined by Traub in [7] as

$$CI = p^{\frac{1}{op}}, \tag{1}$$

where op is the total number of products and quotients per iteration.

In this work, we propose the following family of three-step iterative methods for solving nonlinear systems, obtained by using the technique of undetermined coefficients and the composition of Newton's scheme with itself, with frozen Jacobian. We denote this class by M4 and its iterative expression is

$$\begin{aligned} y_k &= x_k - F'(x_k)^{-1}F(x_k),\\ z_k &= y_k - \frac{1}{\beta}F'(x_k)^{-1}F(y_k),\\ x_{k+1} &= z_k - F'(x_k)^{-1}((2-1/\beta-\beta)F(y_k)+\beta F(z_k)), \end{aligned} \tag{2}$$

where β is an arbitrary real parameter, $\beta \neq 0$. In Section 2, we will prove that all the elements of this family have order of convergence at least four, except for the scheme corresponding to $\beta = 1/5$ that is the unique member of the family with order of convergence five. This element was studied by the authors in [8] where it was applied for solving the problem of preliminary orbit determination of artificial satellites. We will analyze the efficiency of the proposed family by comparing its classical and computational efficiency indices with the corresponding ones of some known schemes of the same order of convergence, the most of them of recent publication.

The stability of the members of family M4 can be studied by applying the usual tools of the complex dynamics. The application of iterative methods for solving nonlinear scalar problems $f(z) = 0$, with $f: \mathbb{C} \to \mathbb{C}$, gives rise to functions whose dynamics are not well-known. The simplest model is obtained when $f(z)$ is a quadratic polynomial and the iterative algorithm is Newton's scheme. This case has been widely studied in the literature (see, for instance [9,10]). The study of the dynamics of Newton's method has been extended to other one-point iterative schemes (see for example [11–14]) and to multipoint iterative methods (see for example [15–20]), for solving nonlinear equations.

From the numerical point of view, the dynamical properties of the rational function associated with an iterative method on polynomials, give us important information about its stability and reliability. In most of the mentioned papers, interesting dynamical planes, including periodical behavior and others anomalies, have been obtained. We are interested in the parameter space associated to family M4 on quadratic polynomials, which allows us to understand the behavior of the different elements of the family, helping us in the election of a particular one with good numerical properties. The qualitative behavior on quadratic polynomials can be extrapolated to more difficult nonlinear equations, that is, an element of the family with bad numerical properties on polynomials will have the same bad behavior on other nonlinear equations.

In this paper, we are going to analyze the dynamics of family (2) when it is applied on quadratic polynomials, characterizing the stability of all the fixed points and studying the critical points of the associated rational function. In this case we consider complex values of the parameter β. The graphic tools used to obtain the parameter space and the different dynamical planes have been introduced by Magreñán in [21] and Chicharro et al. in [22], respectively.

The rest of the paper is organized as follows: in Section 2 we prove the fourth-order of convergence of the elements of family (2), in Section 3 we study the dynamics of family (2) on quadratic polynomials, analyzing the fixed and critical points of the associated operator to the family, $O_p(z, \beta)$, and the stability of the fixed points. The dynamical study of the family is completed in Section 4, by using the associated parameter space and analyzing different regions of variation of parameter β whose corresponding iterative schemes have bad numerical properties. We finish the work with some remarks and conclusions.

2. Convergence analysis

We are going to analyze the local convergence of the methods of family (2). The next result establishes that these schemes have order of convergence at least four, under the standard conditions of the local convergence. In the proof we will use the notation introduced in [23].

Theorem 1. *Let $F: D \subseteq \mathbb{R}^n \to \mathbb{R}^n$, $n \geq 1$, be a sufficiently differentiable function in an open convex set D and let $\bar{x} \in D$ be a solution of the system of nonlinear equations $F(x) = 0$. We suppose that $F'(x)$ is continuous and nonsingular at $\bar{x}$. Then, the sequence $\{x_k\}_{k\geq 0}$ obtained using expression (2) converges to $\bar{x}$ with order of convergence at least four by choosing an initial guess x_0 close enough to the solution. Moreover, the error equation is*

$$e_{k+1} = \left(5 - \frac{1}{\beta}\right)C_2^3 e_k^4 + O(e_k^5),$$

where $C_q = \frac{1}{q!}[F'(\bar{x})]^{-1}F^{(q)}(\bar{x})$, $q \geq 2$, and $e_k = x_k - \bar{x}$.