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Rotations in discrete Clifford analysis

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ABSTRACT

The Laplace and Dirac operators are rotation invariant operators which can be neatly expressed in (continuous) Euclidean Clifford analysis. In this paper, we consider the discrete counterparts of these operators, i.e. the discrete Laplacian Δ or star-Laplacian and the discrete Dirac operator ∂. We explicitly construct rotations operators for both of these differential operators (denoted by $\Omega_{a,b}$ and $dR(e_{a,b})$ respectively) in the discrete Clifford analysis setting. The operators $\Omega_{a,b}$ satisfy the defining relations for $\mathfrak{so}(m,\mathbb{C})$ and they are endomorphisms of the space $\mathcal{H}_k$ of k-homogeneous (discrete) harmonic polynomials, hence expressing $\mathcal{H}_k$ as a finite-dimensional $\mathfrak{so}(m,\mathbb{C})$-representation. Furthermore, the space $\mathcal{M}_k$ of (discrete) k-homogeneous monogenic polynomials can likewise be expressed as $\mathfrak{so}(m,\mathbb{C})$-representation by means of the operators $dR(e_{a,b})$. We will also consider rotations of discrete harmonic (resp. monogenic) distributions, in particular point-distributions, which will allow us to evaluate functions in a rotated point. To make the discrete rotations more visual, we explicitly calculate the rotation of general point-distributions in two dimensions, showing the behavior of such discrete rotations in relation to the continuous case.

1. Introduction

The (massless) Dirac operator finds its origin in particle physics, from the study of elementary particles with spin number $\frac{1}{2}$ [1,2]. This operator is also well-studied in the area of Euclidean Clifford analysis, where it factorizes the Laplace operator, and is hence the corner stone of this theory which refines harmonic analysis. It is well known that both the Laplace and Dirac operator are rotation invariant operators, i.e. invariant under the groups SO(m) and Spin(m) respectively (see e.g. [3]), or equivalently, their mutual Lie algebra $\mathfrak{so}(m)$. From a computer science point of view however, discrete counterparts of both operators are far more interesting, in the sense that they would act on functions defined on a grid $(h\mathbb{Z})^m$ with mesh size h rather than the continuous m-dimensional space $\mathbb{R}^m$.

In recent years, a function-theory studying discrete functions defined on the standard grid $\mathbb{Z}^m$, has arisen as discrete counterpart of the Euclidean Clifford analysis [4,5]. Different approaches vary in the choice of discrete Dirac operator (see e.g. [6–8]), containing forward, backward or central differences. The 'split' discrete Clifford analysis setting (see e.g. [9–11]) introduces a discrete Dirac operator ∂ containing both forward and backward differences $\Delta^{\pm}$, leading to a true factorization ($\Delta^* = \partial^2$) of the standard star-Laplacian with respect to the grid $\mathbb{Z}^m$. At the base of this method lies the concept of the splitting of the basis elements e_i into forward and backward basis elements $\mathbf{e}_i^+$ and $\mathbf{e}_i^-$. The function-theory has shown promising results regarding polynomial solutions, in particular with the definition of discrete monogenic basis elements (Fueter polynomials [12]), a Taylor series decomposition [13], a Cauchy–Kovalevskaya extension theorem [14] and so

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on. It can even be applied to solve differential equations such as the discrete heat equation [15]. However, until now, the representation-theoretical aspects underlying this function-theory, including the rotational invariance of the star-Laplacian and discrete Dirac operator, had not been specified. This will be the topic of this paper.

In classical harmonic analysis, the Laplace operator is a rotation-invariant operator, i.e. invariant under the rotation group $SO(m,\mathbb{C})$ or equivalently, its Lie algebra $\mathfrak{so}(m,\mathbb{C})$. The space of $\mathbb{C}$-valued harmonic polynomials homogeneous of degree k is in fact a model for an irreducible $SO(m,\mathbb{C})$-representation with highest weight $(k,0,\dots,0)$ [16,17]. A similar result is true for spinor-valued monogenic polynomials, homogeneous of degree k, where the highest weight of the irreducible representation is given by $\left(k+\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2}\right)$ in the case of an odd dimension. Since the space of Dirac spinors $\mathbb{S}$ decomposes as direct sum of positive and negative Weyl spinors $\mathbb{S}^+\oplus\mathbb{S}^-$ in even dimension, the space of spinor-valued monogenic polynomials homogeneous of degree k decomposes in even dimension in the sum of exactly two irreducible $SO(m,\mathbb{C})$-representations with highest weights $\left(k+\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2}\right)$ and $\left(k+\frac{1}{2},\frac{1}{2},\dots,\frac{1}{2},-\frac{1}{2}\right)$. In this paper, the aim is to prove similar results in discrete Clifford analysis. In order to do that, we first need to define rotations in the discrete case, for which the star Laplacian Δ^* (resp. discrete Dirac operator ∂) and the degree of homogeneity (i.e. the Euler operator) are invariant. Moreover, the found operators should generate the Lie algebra $\mathfrak{so}(m,\mathbb{C})$. The operators involved, $L_{a,b}\cdot=(\xi_a\,\partial_b+\xi_b\,\partial_a)\cdot e_b e_a$ resp. $dR(e_{a,b})\cdot=e_a\,e_b\,e_a^\perp\,e_b^\perp\left(L_{a,b}-\frac{1}{2}\right)\cdot e_a^\perp\,e_b^\perp$, may seem to have the same definition or structure as in the classical case [17–19], however, the added basis elements (both from the left and from the right) will have a grave impact when considering irreducible modules.

Although it may abstractly be seen as 'just' another realization of the Lie algebra $\mathfrak{so}(m,\mathbb{C})$, the introduction of these operators makes it possible to rotate discrete points, meaning points of the grid $\mathbb{Z}^m$, over an arbitrary angle (not necessarily an integer multiple of $\frac{\pi}{2}$) without leaving the grid, instead spreading the original point into a cloud of grid-points (see Section 8). It makes clear that, although we restricted the points to the grid, the rotations are also inherently present in the discrete Clifford analysis setting. The problem that a rotated discrete point does not necessarily is concentrated in a single grid-point, has been made irrelevant by allowing such a rotated point to be split (in an intrinsic way) over nearby grid-points. Of course, when projecting on the classic (Euclidean) Clifford algebra, all results must be in accordance to the corresponding results.

To keep the paper self-contained we include Section 2, containing the details of the discrete split Clifford analysis section. In Section 3, we define the visual projection, a tool to desplit our discrete Clifford algebra and make the connection with the classical continuous Clifford algebra. We will first introduce the discrete rotations $\Omega_{a,\,b}$ for the space of discrete harmonic functions, in Section 4, and show that these not only generate $\mathfrak{so}(m,\mathbb{C})$, but also commute with the Laplacian Δ, its symbol ξ^2 and the corresponding Euler operator $\mathbb{E}$. Next, in Section 5, we consider the space of monogenic functions where we introduce the endomorphisms $dR(e_{a,\,b})$, generating $\mathfrak{so}(m,\mathbb{C})$ and commuting with the discrete Dirac operator ∂, its symbol ξ and $\mathbb{E}$. In both sections, the corresponding Casimir operator is briefly considered. We then extend the notion of discrete rotations to the dual spaces $\mathcal{H}'$ and $\mathcal{M}'$ of discrete harmonic and monogenic distributions in Sections 6 and 7. The most important section is Section 8, where we explicitly calculate the general formula for the rotation of a discrete point-distribution in two dimensions. It shows how a discrete point (in two dimensions) is rotated to a cloud of grid-points for a general angle, while this cloud recombines to a single grid-point if the angle represents a well-defined rotation on the grid.

2. Preliminaries

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space with orthonormal basis e_j, $j=1,\dots,m$ and consider the Clifford algebra $\mathbb{R}_{m,0}$ over $\mathbb{R}^m$. Passing to the so-called discrete 'split' Clifford setting, we embed the Clifford algebra $\mathbb{R}_{m,0}$ into the bigger complex one $\mathbb{C}_{2m}$, i.e. the underlying vector space is of twice the dimension. This is done by adding another set of generators $e_j^\perp$, $j=1,\dots,m$ satisfying the relations

$$e_je_\ell+e_\ell e_j=2\delta_{j\ell},\quad e_j^\perp e_\ell^\perp+e_\ell^\perp e_j^\perp=-2\delta_{j\ell},\quad e_je_\ell^\perp+e_\ell^\perp e_j=0,$$

for all $j,\ell=1,\dots,m$. In this algebra, we introduce forward and backward basis elements $\mathbf{e}_j^\pm=\frac{e_j\pm e_j^\perp}{2}$, satisfying the following anti-commutator rules, cf. [10]:

$$\mathbf{e}_j^-\mathbf{e}_\ell^-+\mathbf{e}_\ell^-\mathbf{e}_j^-=\mathbf{e}_j^+\mathbf{e}_\ell^++\mathbf{e}_\ell^+\mathbf{e}_j^+=0 \text{ and } \mathbf{e}_j^+\mathbf{e}_\ell^-+\mathbf{e}_j^-\mathbf{e}_\ell^+=\delta_{j\ell}.$$

The connection to the original basis e_j is hence given by $\mathbf{e}_j^++\mathbf{e}_j^-=e_j$. Observe that we consider $e_j^2=1$ in contrast to the usual Clifford setting where traditionally $e_j^2=-1$ is chosen.

This approach is motivated due to the aim of constructing a discrete Dirac operator factorizing the discrete (star-)Laplacian. Consider therefore the standard equidistant lattice $\mathbb{Z}^m$, i.e. the coordinates of a Clifford vector $\underline{x}$ take only integer values. The partial derivatives ∂_{x_j} used in Euclidean Clifford analysis are replaced by forward and backward differences $\Delta_j^\pm$, $j=1,\dots,m$, acting on Clifford-valued functions f as follows:

$$\Delta_j^+[f](\cdot)=f(\cdot+e_j)-f(\cdot),\quad \Delta_j^-[f](\cdot)=f(\cdot)-f(\cdot-e_j).$$