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Defect correction finite element method for the stationary incompressible Magnetohydrodynamics equation

Zhiyong Si[a,*], Shujie Jing[a,b], Yunxia Wang[c]

[a] School of Mathematics and Information Science, Henan Polytechnic University, Jiaozuo 454003, PR China
[b] School of Mathematic Science, Dalian University of Technology, Linggong Road, Dalian, Liaoning Province 116024, PR China
[c] School of Materials Science and Engineering, Henan Polytechnic University, Jiaozuo 454003, PR China

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ABSTRACT

In this study, we give the defect correction finite element method for the stationary incompressible MHD equation. Firstly, the nonlinear MHD equation is solved with an artificial viscosity term. Then, the numerical solution is improved on the same grid by a linearized defect-correction technique. Then, we give the numerical analysis including stability analysis and error analysis. The numerical analysis proves that our method is stable and has an optimal convergence rate. Then, we give some numerical results. From the numerical results, we can see that our method is efficient for solving the MHD equations.

1. Introduction

In this paper, we study the stationary incompressible magnetohydrodynamics (MHD) equation, which has been applications in fusion technology and novel submarine propulsion devices. The MHD equation is a coupled equation system of Navier–Stokes equations of fluid dynamics and Maxwell's equations of electromagnetism via Lorentz's force and Ohm's Law. This equation also models the flow of liquid metals in magnetic pumps that are used to cool nuclear reactors [15,20,34]. It was used in many branches of physics and engineering technology, such as wave propagation in ionosphere in geophysics; MHD engine; control of MHD boundary layer and revival of liquid-metal MHD electricity generation.

In the past decades, there are many numerical approaches for solving the MHD equation. In [35], a finite element analysis of two dimensional MHD flow was given. Gunzburger et al. [15] gave a mixed finite element method for MHD equation. In [13,16], a mixed finite element method with exactly divergence-free velocities was studied in non-convex polygonal/polyhedral domain. Recently, the streamline diffusion finite element method for stationary incompressible MHD equations was given [37]. A convergence analysis of three finite element iterative methods for the 2D/3D stationary incompressible MHD equations was presented by Dong et al. [6]. Author of [17] gave the unconditional convergence of the Euler semi-implicit scheme for the three-dimensional incompressible MHD equation. A two level Newton iterative method for the 2D/3D stationary incompressible MHD equation was given by Dong and He [7]. Analysis of coupling iterations based on the finite element method for stationary MHD equation was shown in [36]. The authors of [28] gave the finite volume spectral element method for solving magnetohydrodynamic (MHD) equations. In [19], the boundary elements method

* Corresponding author. Tel.: +86 15938183695.
E-mail addresses: zhiyongsi@gmail.com, sizhiyong@hpu.edu.cn (Z. Si), jsj_jjj@hpu.edu.cn (S. Jing), wangyunxia@hpu.edu.cn (Y. Wang).

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for magneto-hydrodynamic (MHD) channel flows at high Hartmann numbers was given. In [2–5], Dehghan et al. gave the meshless method for solving the unsteady magnetohydrodynamic (MHD) flow.

In the defect-correction method, the nonlinear equations with an added artificial viscosity term are solved and this solution is corrected on the same grid by using a defect-correction technique. In fact, it is an iterative improvement technique for increasing the accuracy of a numerical solution without applying a grid refinement. Due to its good efficiency, there are many works devoted to this method, e.g. the convection–diffusion equation [8], adaptive refinement for the convection–diffusion problems [1], adaptive defect correction methods for the viscous incompressible flow [9], two-parameter defect-correction method for computation of the steady-state viscoelastic fluid flow [10], variational methods for the elliptic boundary value problems [11], defect-correction parameter-uniform numerical method for a singularly perturbed convection–diffusion problem [12], finite volume local defect correction method for solving the transport equation [21], the singular initial value problems [22], the time-dependent Navier–Stokes equations [23,26], the stationary Navier–Stokes equation [24], second order defect correction scheme [25] and so on. A method for solving the time dependent Navier-Stokes equations, aiming at high Reynolds number, was presented in [23]. The authors of [31–33] have given the defect correction finite element method for the conduction–convection problems. We also give the defect correction method combined with modified characteristics method for time dependent Navier–Stokes equations [29,30].

In the paper, we give the defect correction finite element method (FEM) for the stationary incompressible MHD equation. Firstly, the nonlinear MHD equation is solved with an artificial viscosity term. Then, the numerical solution on the same grid by a linearized defect-correction technique. Actually, the defect-correction finite element method incorporates the artificial viscosity term as a stabilizing factor, making both the nonlinear system easier to be solved and the linearized system easier to be preconditioned. Then, we give the numerical analysis including stability analysis and error analysis. The numerical analysis proves that our method is stable and has an optimal convergence rate. At last, we give some numerical results. In the numerical experiments, we solve the nonlinear MHD equation by an iterative method with an artificial viscosity term. The numerical results were improved by the defect-correction method. From the numerical results, we can see that our method is efficient for solving the MHD equations.

2. Functional settings of the stationary MHD equation

Let $\Omega \subset \mathbb{R}^d (d = 2, 3)$ be a bounded domain. As in [15], we assume that Ω is a convex polygonal/ployhedral domain in $\mathbb{R}^d (d = 2, 3)$. In this paper, we consider the stationary incompressible MHD equations as follows

$$\begin{cases} -R_e^{-1}\Delta \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - S_c \mathrm{curl}\mathbf{B}\times\mathbf{B} = f, & x \in \Omega, \\ S_c R_m^{-1}\mathrm{curl}(\mathrm{curl}\mathbf{B}) - S_c \mathrm{curl}(\mathbf{u}\times\mathbf{B}) = g. & x \in \Omega, \\ \nabla\cdot\mathbf{u} = 0, & x \in \Omega, \\ \nabla\cdot\mathbf{B} = 0, & x \in \Omega, \end{cases} \tag{2.1}$$

with homogeneous boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \quad \text{(no-split condition)}$$
$$(\mathbf{B}\cdot\mathbf{n})|_{\partial\Omega} = 0, \quad (\mathbf{n}\times\mathrm{curl}\,\mathbf{B})|_{\partial\Omega} = 0, \quad \text{(perfectly conducting wall)},$$

where $\mathbf{u}$ represents the velocity of the fluid flow, $\mathbf{B}$ represents magnetic field, p represents the pressure, f and g represent the external body force terms. The MHD equation is characterized by three parameters: R_e hydrodynamic Reynolds number, R_m magnetic Reynolds number and the coupling number S_c.

In this paper, we employ the stand scalar Sobolev space $H^k(\Omega) = W^{k,2}(\Omega)$ for nonnegative k, equipped with the norm $\|v\|_k = (\sum_{|\alpha|=0}^{k}\|D^\alpha v\|_0^2)^{1/2}$. For vector-value functions, we use the Sobolev space $\mathbf{H}^k(\Omega) = (H^k(\Omega))^d$ with the norm $\|\mathbf{v}\|_k = (\sum_{i=1}^{d}\|v_i\|_k^2)^{1/2}$ (see [14] for details). We set

$$\begin{aligned} X &:= \mathbf{H}_0^1(\Omega) = \{\mathbf{v}\in\mathbf{H}^1(\Omega), \mathbf{v}|_{\partial\Omega} = 0\}, \\ W &:= \mathbf{H}_n^1(\Omega) = \{\mathbf{v}\in\mathbf{H}^1(\Omega), \mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0\}, \\ V &:= \{\mathbf{v}\in X, \nabla\cdot\mathbf{v} = 0\}, \\ V_n &:= \{\mathbf{v}\in W, \nabla\cdot\mathbf{v} = 0\}, \\ M &:= L_0^2(\Omega) = \left\{q\in L^2(\Omega) : \int_\Omega q dx = 0\right\}. \end{aligned}$$

We use the equivalent norm $\|\mathbf{w}\|_1 = \|\nabla\mathbf{w}\|_0$ of X, and employ the product space $W_{0n} := X\times W$.

In addition, the following two formulas of vector is useful in our analysis

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c}\cdot\mathbf{d} = (\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = -(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{d}\cdot\mathbf{c}), \tag{2.2}$$

and

$$\int_\Omega (\nabla\times\Phi)\cdot\Psi dx = -\int_{\partial\Omega}(\Phi\times\mathbf{n})\cdot\Psi dx + \int_\Omega \Phi\cdot(\nabla\times\Psi)dx. \tag{2.3}$$