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A weak finite element method for elliptic problems in one space dimension

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ABSTRACT

We present a weak finite element method for elliptic problems in one space dimension. Our analysis shows that this method has more advantages than the known weak Galerkin method proposed for multi-dimensional problems, for example, it has higher accuracy and the derived discrete equations can be solved locally, element by element. We derive the optimal error estimates in the discrete H^1-norm, the L_2-norm and L_∞-norm, respectively. Moreover, some superconvergence results are also given. Finally, numerical examples are provided to illustrate our theoretical analysis.

1. Introduction

Recently, the weak Galerkin finite element method attracts much attention in the field of numerical partial differential equations [1–6]. This method is presented originally by Wang and Ye for solving elliptic problem in multi-dimensional domain [1]. Since then, some modified weak Galerkin methods have also been studied, for example, see [7–10]. The weak Galerkin method can be considered as an extension of the standard finite element method where classical derivatives are replaced in the variational equation by the weak derivatives defined on weak finite element functions. The main feature of this method is that it allows the use of totally discontinuous finite element function and the value of finite element function on element boundary may be independent with its value in the interior of element. This feature makes this method possess the advantage of the usual discontinuous Galerkin (DG) finite element method [11–13] and it has higher flexibility than the DG method. The readers are referred to articles [2,3,12] for more detailed explanation of this method and its relation with other finite element methods.

In this paper, we present a weak finite element method for general second order elliptic problem in one space dimension:

$$\begin{cases} -(a_2(x)u')' + a_1(x)u' + a_0(x)u = f(x), \quad x \in (a, b), \\ u(a) = 0,\ u'(b) = 0, \end{cases} \tag{1.1}$$

where $a_2(x) \geq a_{min} > 0, a_0(x) \geq 0$.

We first define the weak derivative and discrete weak derivative on discontinuous function in one dimensional domain. Then, we construct the weak finite element space S_h and use it to give the weak finite element approximation to problem (1.1). Although, in some aspects, our method uses the idea of the original weak Galerkin finite element method proposed

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for multi-dimensional problem [1], it still has itself features. For example, our weak finite element space S_h admits a weak embedding inequality (see Lemma 3.2), which can be used to derive the L_∞-error estimate on mesh point set; Next, the discrete finite element system of equations can be solved locally, element by element. These features are not available for the weak Galerkin method in multi-dimensional space. Except the usual optimal error estimates in various norms, we also give some surperconvergence results for the weak finite element solution. Numerical results show that our method possesses very high computation accuracy. For finite element polynomial of order k, our computations show that the numerical convergence rates are at least of order $k+2$ in the discrete H^1-norm, the L_2-norm and the L_∞-norm at mesh points. Our method also can be applied to solve other partial differential equations in one space dimension.

This paper is organized as follows. In Section 2, we introduce the weak finite element method for the elliptic problem. In Section 3, the stability of the weak finite element method is analyzed. Section 4 is devoted to the optimal error estimate and superconvergence estimate in various norms. In Section 5, the local solvability of the weak finite element system of equations is discussed and numerical experiments are provided to illustrate our theoretical analysis.

Throughout this paper, we adopt the notations $H^m(I)$ to indicate the usual Sobolev spaces on interval I equipped with the norm $\|\cdot\|_m = \|\cdot\|_{H^m(I)}$. The notations $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and norm, respectively, in the space $L_2(I)$. We will use letter C to represent a generic positive constant, independent of the mesh size h.

2. Problem and its weak finite element approximation

Consider elliptic problem (1.1). Multiplying Eq. (1.1) by the transformation function

$$\rho(x) = exp\Big(-\int_0^x \frac{a_1(x)}{a_2(x)}dx\Big),$$

we see that problem (1.1) can be transformed into the following form:

$$-(\rho\, a_2 u')' + \rho\, a_0 u = \rho\, f(x), \quad x \in (a,b), \quad u(a)=0, \quad u'(b)=0.$$

Therefore, in what follows, we only consider elliptic problems in the form:

$$\begin{cases} -(a_2(x)u')' + a_0(x)u = f(x), & x \in (a,b), \\ u(a)=0,\ u'(b)=0, \end{cases} \tag{2.1}$$

where $a_2(x) \geq a_{min} > 0, a_0(x) \geq 0$ and $u' = \frac{du}{dx}$. We assume that $a_2(x) \in H^1(a,b), a_0(x) \in L_\infty(a,b)$.

First, let us introduce the weak derivative concept. Let closed interval $\bar{I}_a = [x_a, x_b]$ and its interior $I_a = (x_a, x_b)$. A weak function on $\bar{I}_a$ refers to a function $v = \{v^0, v^a, v^b\}$, $v^0 = v|_{I_a} \in L_2(I_a)$, values $v^a = v(x_a)$ and $v^b = v(x_b)$ exist. Note that v^a and v^b may not be necessarily the trace of v^0 at the interval endpoints x_a and x_b. Denote the weak function space by

$$W(I_a) = \{v = \{v^0, v^a, v^b\} : v^0 \in L_2(I_a), |v^a| + |v^b| < \infty\}.$$

Definition 2.1. Let $v \in W(I_a)$. The weak derivative $d_w v$ of v is defined as a linear functional in the dual space $H^{-1}(I_a)$ whose action on each $q \in H^1(I_a)$ is given by

$$< d_w v, q > \doteq -\int_{I_a} v^0 q' dx + v^b q^b - v^a q^a, \quad \forall q \in H^1(I_a), \tag{2.2}$$

where $q^a = q(x_a), q^b = q(x_b)$.

Obviously, as a bounded linear functional on $H^1(I_a)$, $d_w v$ is well defined for any $v \in W(I_a)$. Moreover, for $v \in H^1(I_a)$, if we consider v as a weak function with components $v^0 = v|_{I_a}$, $v^a = v(x_a)$ and $v^b = v(x_b)$, then by integration by parts, we have for $q \in H^1(I_a)$ that

$$\int_{I_a} v' q dx = -\int_{I_a} v q' dx + v^b q^b - v^a q^a = -\int_{I_a} v^0 q' dx + v^b q^b - v^a q^a, \tag{2.3}$$

which implies that $d_w v = v'$ is the usual derivative of function v if $v \in H^1(I_a)$.

Next, we introduce the discrete weak derivative which is actually used in our analysis. For nonnegative integer $r \geq 0$, let $P_r(I_a)$ be the space composed of all polynomials on I_a with degree no more than r. Then, $P_r(I_a)$ is a subspace space of $H^1(I_a)$.

Definition 2.2. For $v \in W(I_a)$, the discrete weak derivative $d_{w,r} v \in P_r(I_a)$ is defined as the unique solution of the following equation

$$\int_{I_a} d_{w,r} v q dx = -\int_{I_a} v^0 q' dx + v^b q^b - v^a q^a, \quad \forall q \in P_r(I_a). \tag{2.4}$$

Let $v \in H^1(I_a)$. From (2.3) and (2.4), we have

$$\int_{I_a} (d_{w,r} v - v') q dx = 0, \quad \forall q \in P_r(I_a).$$

This implies that $d_{w,r} v$ is the L_2 projection of v' in $P_r(I_a)$ if $v \in H^1(I_a)$.