Applied Mathematics and Computation 280 (2016) 46–56

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

Numerical methods for solving two-dimensional nonlinear integral equations of fractional order by using two-dimensional block pulse operational matrix

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ARTICLE INFO

Keywords:
Two-dimensional fractional integral equations
Two-dimensional block pulse functions
Fractional operational matrix

ABSTRACT

In this paper, our purpose is to construct a two-dimensional fractional integral operational matrix and its use for the numerical solution of two-dimensional fractional integral equations. We use these operational matrices and properties of two-dimensional block pulse functions (2D-BPFs), to reduce two-dimensional fractional integral equations (2D-FIEs) to a system of algebraic equations. Obtained algebraic system based on the original problem can be linear or nonlinear. Then we show convergence of the proposed methods and we find the error bounds. To show the accuracy, efficiency and speed of the proposed method linear and nonlinear examples are presented.

1. Introduction

Fractional calculus story dates back to 1965 by a letter which L'Hopital wrote to Leibniz and with passage of time its complexity and applications increased [1,2]. Numerous applications in various branches of science, including physics, engineering and mathematics have been proposed for it by many studies carried out in last years. For example in [3] Podlubny uses the Laplace transform for solving linear differential equations of fractional order. Jafari and Momani used homotopy perturbation method for solving fractional diffusion and wave equations [4]. In [5] authors used Walsh operational matrices for fractional calculus. Also Laguerre operational matrices were used by Hwang and Shih for fractional calculus [6]. Fractional nonlinear differential equations were solved by Adomian decomposition method [7]. Khader in [8] presented numerical method for multi-order fractional differential equations by using operational matrix. In [9], inverse Laplace transform algorithms were used for numerical method in fractional calculus. Ma and Huang use a hybrid collocation method for integro-differential equations of fractional order [10]. The last example in [11] spline collocation methods were used for solving fractional differential equations.

Two-dimensional fractional equations are important subdivision of fractional calculus that we introduce in this paper. In the following we briefly review the works done in this field. Homotopy perturbation method is used for two dimensional time-fractional wave equation [12]. In [13,14] two-dimensional fractional percolation equation was solved. Orthogonal spline collocation method was used for the two-dimensional fractional sub-diffusion equation [15]. In [16] we see an applied problem and analytical approximate method for solving it in two and three dimensions.

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http://dx.doi.org/10.1016/j.amc.2015.12.042
0096-3003/

Block pulse functions have already been used for solving fractional equations which are going to be discussed later in this paper. Yi et al. in [17] presented block pulse operational matrix method for solving fractional partial differential equation, also in [18] this method was used for fractional differential equations by Li and Sun. In addition, in [19,20] authors used block pulse functions (BPFs) for solving fractional order equations.

In this paper first we will briefly review fractional calculus. Then we introduce (BPFs), (2D-BPFs) and some properties of them. Afterward we present approximate function via (2D-BPFs) and we use this approximate and integration of (2D-BPFs) to find operational matrix of two-dimensional integration of fractional order. In Section 4 using operational matrix we can reduce two-dimensional nonlinear integral equations of fractional order to a system of algebraic equations. Newton's iterative method can be used for solving nonlinear algebraic system. Finally, we present the convergence of the proposed methods and we find the error bounds. In Section 6 with some numerical examples we show accuracy and efficiency of this method.

2. Brief review of fractional integrals

In this section, first we present a short introduction of fractional calculus used in this paper and then we express properties about them.

Definition 2.1. The Rieman–Liouville fractional integral operator I^α of order α on the usual Lebesgue space $L_1[a, b]$ is given by

$$(I^\alpha f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau, \quad for \quad \alpha > 0, \tag{2.1}$$

where $t > 0$ and is a real positive order, and $\Gamma(.)$ is the Euler gamma function:

$$\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} dx. \tag{2.2}$$

Some properties of Rieman–Liouville fractional integral are the following:

(1) $(I^0 f)(t) = f(t),$

(2) $(I^\alpha I^\beta f)(t) = (I^\beta I^\alpha f)(t) = (I^{\alpha+\beta} f)(t),$

(3) $$I^\alpha (t-a)^n = \frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)} (t-a)^{n+\alpha}. \tag{2.3}$$

Similarly, with let $J := [0, a] \times [0, b]$ and $L^1(J)$ is the space of Lebesgue-integrable functions $w : J \longrightarrow R^n$ with the norm $\| w \|_{L^1} = \int_0^a \int_0^b \| w(x, y) \| dydx$, we have the following definition for the two dimensional case.

Definition 2.2. The left-sided mixed Riemann–Liouville integral of order r of u is defined as [21]

$$(I_\theta^r u)(x, y) = \frac{1}{\Gamma(r_1)\Gamma(r_2)} \int_0^x \int_0^y (x-s)^{r_1-1} (y-t)^{r_2-1} u(s, t) dtds \tag{2.4}$$

where $r = (r_1, r_2) \in (0, \infty) \times (0, \infty),\ \theta = (0, 0)$ and $u \in L^1(J)$.

According to Definition 2.2 we have the following properties of left-sided mixed Riemann–Liouville integral:

(1) $(I_\theta^\theta u)(x, y) = u(x, y),$

(2) $If\ \sigma = (1, 1)\ then,\ (I_\theta^\sigma u)(x, y) = \int_0^x \int_0^y u(x, y) dtds, \quad for\ all\ (x, y) \in J,$

(3) $(I_\theta^r u)(x, 0) = (I_\theta^r u)(0, y) = 0, \quad for\ x \in [0, a], y \in [0, b],$

(4) $$Let\ \lambda, \omega \in (-1, \infty)\ then,\ I_\theta^r x^\lambda y^\omega = \frac{\Gamma(\lambda+1)\Gamma(\omega+1)}{\Gamma(\lambda+r_1+1)\Gamma(\omega+r_2+1)} x^{\lambda+r_1} y^{\omega+r_2}, \quad for\ all\ (x, y) \in J. \tag{2.5}$$

For more information of left-sided mixed Riemann–Liouville integral see [21].