Applied Mathematics and Computation 280 (2016) 103–109

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# The extremal values of some topological indices in bipartite graphs with a given matching number☆

Hanlin Chen, Renfang Wu, Hanyuan Deng*

*College of Mathematics and Computer Science, Hunan Normal University, Changsha, Hunan 410081, PR China*

ARTICLE INFO

*MSC:*
05C07
05C15
05C50

*Keywords:*
Topological indices
Bipartite graphs
Matching number

ABSTRACT

Let $I(G)$ be a topological index of a graph. If $I(G+e) < I(G)$ (or $I(G+e) > I(G)$, respectively) for each edge $e \notin G$, then $I(G)$ decreases (or increases, respectively) with addition of edges. In this paper, we determine the extremal values of some topological indices which decrease or increase with addition of edges, and characterize the corresponding extremal graphs in bipartite graphs with a given matching number.



## 1. Introduction

Topological indices are numerical parameters of a graph which characterize its topology and are usually graph invariants. Examples include the number of vertices and the number of edges, the Zagreb index, the Wiener index, the Hosoya index and the matching energy etc.

All graphs considered in this paper are simple and connected. Let $G$ be a graph with vertex and edge sets $V(G)$ and $E(G)$, respectively. For $u \in V(G)$, $d(u)$ denotes the degree of $u$ in $G$. The distance $d(u, v)$ is defined as the length of a shortest path connecting $u$ and $v$.

A matching in a graph is a set of pairwise nonadjacent edges. If $M$ is a matching, the two ends of each edge of $M$ are said to be matched under $M$, and each vertex incident with an edge of $M$ is said to be an $M$-saturated. A perfect matching is one which saturates every vertex of the graph, a maximum matching is one which saturates as many vertices as possible. Recall that the number of edges in a maximum matching of a graph $G$ is called the matching number of $G$ and denoted by $\beta(G)$.

Let $M$ be a matching in a graph $G$. An $M$-alternating path or cycle in $G$ is a path or cycle whose edges are alternately in $M$ and $E - M$. An $M$-alternating path might or might not start or end with edges of $M$. If neither its origin nor its terminus is saturated by $M$, the path is called an $M$-augmenting path. The Berge's theorem [1] points out the relevance of augmenting paths to the study of maximum matchings, i.e., a matching $M$ in a graph $G$ is a maximum matching if and only if $G$ contains no $M$-augmenting path.

☆ Project supported by the National Natural Science Foundation of China (61572190) and Hunan Provincial Innovation Foundation for Postgraduate (CX2015B162).

* Corresponding author. Tel.: +86 073188872516.
*E-mail address:* hydeng@hunnu.edu.cn (H. Deng).

http://dx.doi.org/10.1016/j.amc.2016.01.042
0096-3003/

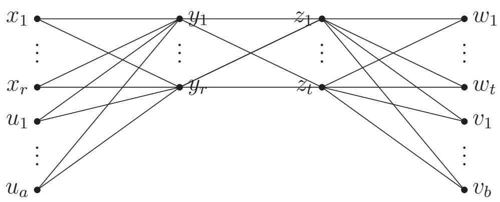


**Fig. 1.** $B(r, t, a, b)$.

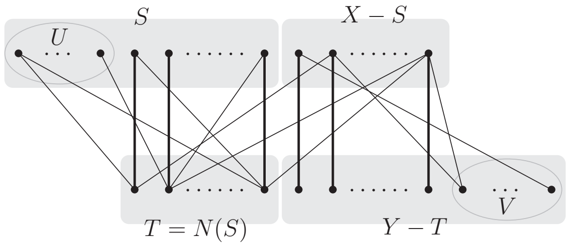


**Fig. 2.** The graph for the proof of Proposition 1.

## 2. Basic properties

We consider the effect of edge addition (or deletion) on topological indices.

Let $I(G)$ be a topological index of a graph. If $I(G+e) < I(G)$ (or $I(G+e) > I(G)$, respectively) for each edge $e \notin G$, then $I(G)$ decreases (or increases, respectively) with addition of edges.

For example, the Wiener index decreases with addition of edges, the Zagreb index, the Hosoya index and the matching energy increase with addition of edges.

In this section, we first discuss the property of the topological indices decrease/increase with addition of edges among all bipartite graphs of order $n$ with a given matching number $\beta$.

The following expression for the matching number is known as the König–Ore formula. The matching number $\beta$ of a bipartite graph $G$ with bipartition $(X, Y)$ is given by:

$$\beta = |X| - \max\{|S| - |N(S)| : S \subseteq X\}. \tag{1}$$

Let $B(r, t, a, b)$ be a bipartite graph obtained from two complete bipartite graphs $G_1 = K_{r,r+a}$ and $G_2 = K_{t,t+b}$ by joining each vertex of $r$-part in $G_1$ to each vertex of $t$-part in $G_2$, where $r, t, a, b$ are non-negative integers, see Fig. 1.

**Proposition 1.** *Let $G$ be a bipartite graph of order $n$ with bipartition $(X, Y)$ and matching number $\beta$. Then there exists a bipartite graph $G' = B(r, t, a, b)$ such that*

(i) *$I(G') \le I(G)$ for the topological index $I$ which decreases with addition of edges;*
(ii) *$I(G') \ge I(G)$ for the topological index $I$ which increases with addition of edges, with equality if and only if $G \cong B(r, t, a, b)$ for some non-negative integers $r, t, a, b$, where $r + t = \beta$, $r + t + a = |X|$ and $r + t + b = |Y|$.*

**Proof.** Let $M^*$ be a maximum matching of $G$. Denote by $U$ the set of $M^*$-unsaturated vertices in $X$, and by $Z$ the set of all vertices reachable from vertices of $U$ by $M^*$-alternating paths. Set $S = Z \cap X$ and $T = Z \cap Y$. Then we have that every vertex in $T$ is $M^*$-saturated and $N(S) = T$ and there is no edge joining a vertex of $S$ to a vertex of $Y - T$, see Fig. 2. Let $r = |T|$, $t = |X - S|$, $a = |U|$ and $b = |Y| - r - t$. Then $G$ is a spanning subgraph of $G' = B(r, t, a, b)$. So, we have that (i) $I(G') \le I(G)$ for the topological index $I$ which decreases with addition of edges, and (ii) $I(G') \ge I(G)$ for the topological index $I$ which increases with addition of edges, with equality if and only if $G = G'$. □

## 3. Applications

### *3.1. The maximum values of the Hosoya index, the matching energy in bipartite graphs with a given matching number*

In this section, we will give the maximum values of the Hosoya index and the matching energy of bipartite graphs with a given matching number. Some recent results on the Hosoya index and the matching energy can be found in [4–6,10,11,14,27,35].

By $m(G; k)$ we denote the number of $k$-matchings (i.e., the number of selections of $k$ independent edges) of the graph $G$. Specifically, $m(G; 0) = 1$, $m(G; 1) = |E(G)|$ and $m(G; k) = 0$ for $k > \frac{n}{2}$, where $n = |V(G)|$ is the number of vertices in $G$.