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Construction of high-order Runge–Kutta methods which preserve delay-dependent stability of DDEs[☆]

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ABSTRACT

This paper is concerned with the construction of some high-order Runge–Kutta methods, which preserve delay-dependent stability of delay differential equations. The methods of the first kind are developed by extending the ideas of Brugnano et al., while the methods of the second kind are developed according to the structure of the stability matrix. We show that the derived methods are $\tau(0)$-stable for delay differential equations. Meanwhile, the Runge–Kutta methods can own the same order of the accuracy as the Radau methods or Gauss methods if the parameters are adequately defined. These results not only improve the order of accuracy of the methods investigated by Huang, but also open an interesting route of finding new $\tau(0)$-stable Runge–Kutta methods. Finally, numerical experiments are proposed to illustrate the theoretical results.

1. Introduction

Delay differential equations (DDEs) have attracted a significant interest in the last several decades due to their frequent appearance in actual applications [1,6,11,29]. They are usually proposed as models, when the rate of variation does not only depend on the present situation but also on the history. Up to now, many dynamic properties of the solutions of DDEs have been extensively explored. Most results in the literature indicate that a delay term has an important impact on the dynamic properties of a system such as stability [13,19,20,22,23], dissipativity [17], chaos [24], etc.

As a typical test equation in this field, we refer to the following discrete delay differential equation:

$$y'(t) = \mu y(t) + \nu y(t-\tau), \tag{1.1}$$

where $\mu, \nu \in \mathbb{R}, \tau \in \mathbb{R}^+$. The zero solution of Eq. (1.1) is asymptotically stable if and only if $(\mu, \nu) \in S_*^\tau$ (see e.g., [11,18]), where

$$S_*^\tau = \left\{ (\mu, \nu) : \mu\tau < 1 \text{ and } \mu\tau < -\nu\tau < \frac{\theta}{\sin\theta} \right\},$$

where θ is the root of $\theta \cos\theta = \mu\tau \sin\theta$, $\theta \in (0, \pi)$. The stability of Eq. (1.1) depends not only on the coefficients μ and ν, but also on the delay term τ. This is so called delay-dependent stability. Eq. (1.1) is usually used as a test model to identify

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whether a numerical method is effective for solving DDEs. When S_*^τ is the subset of the stability regions of a numerical method for DDEs (1.1), the numerical scheme is called $\tau(0)$-stable (see [12]) or method which can preserve delay-dependent stability of DDEs.

As pointed by Bellen and Zennaro [1], the study of the delay-dependent stability of a numerical method is believed as one of the most challenging problems in the field of DDEs. One important reason is that, compared to the delay independent stability analysis, the delay-dependent stability region is larger and more complicated to describe (cf. [18], also see [1]). In [14], the delay-dependent stability regions of θ-methods for (1.1) were studied. Guglielmi pointed out that if $\frac{1}{2} \le \theta \le 1$, the methods were $\tau(0)$-stable. After that, Guglielmi and Hairer [15] proved that the Gauss and the 2-stage and 3-stage Radau methods were $\tau(0)$-stable. Recently, Huang [18] further proved that all the Radau methods were $\tau(0)$-stable. For more stability analysis of a numerical method for Eq. (1.1) and the other types of DDEs, we refer the readers to the books [1,6], the papers [10,12,21,28,30,32–35] and the references therein.

Up to now, the main emphasis is placed on the delay-dependent stability of some numerical methods for DDEs. Little research has been conducted in developing new methods, which own the same advantage as the Gauss or Radau methods for DDEs. A major thrust of the paper is to discuss approaches and strategies for constructing some new high-order $\tau(0)$-stable numerical methods. The method in this study is a refinement of the recent ideas of Brugnano et al. [2] and Huang [18], but the main results are different. We will explain the differences after discussing stability and convergence of the methods.

Observing the Radau methods and Gauss methods, we find that both the methods can be derived from the *W*-transformation. This fact may offer an effective way to construct $\tau(0)$-stable Runge–Kutta methods for DDEs. As is well known, the W-transformation was first proposed by Hairer and Wanner to construct high-order implicit Runge–Kutta methods. Later, it was used to develop some symplectic Runge–Kutta methods [3–5,16,25–27]. More recently, Brugnano et al. [2] introduced a perturbation in the W-transformation and derived a new family of parametric symplectic integrators based on Gauss methods. In this paper, the new parametric Gauss methods are proved to be $\tau(0)$-stable. Moreover, inspired by Brugnano et al., we develop some parametric Runge–Kutta methods based on some new perturbation matrices. The derived Runge–Kutta methods, depending on a parameter α, have the following properties:

- Under suitable assumptions, the parametric Runge–Kutta methods are $\tau(0)$-stable for DDEs.
- For $\alpha = 0$, the corresponding methods reduce to the classical s-stage Radau method (of order $2s - 1$) or Gauss methods (of order 2s).
- For $\alpha \neq 0$, we have some new parametric Runge–Kutta methods. The s-stage methods can be of order $2s - 2$, $2s - 1$ or $2s$.

The present work not only opens an interesting route of obtaining new $\tau(0)$-stable Runge–Kutta methods by perturbed methods, but also improves the order of accuracy of some $\tau(0)$-stable methods investigated in [18]. Numerical experiments are proposed at last to illustrate the convergence and stability of the methods.

The rest of the paper is organized as follows. In Section 2, some preliminaries for the construction of the methods are proposed. Section 3 describes in detail the construction of the methods. Section 4 is devoted to study the delay-dependent stability of the parametric Runge–Kutta methods for solving DDEs. Section 5 discusses the order of accuracy. Section 6 is developed to compare our results with the existing ones. Section 7 shows experimental studies for verifying the proposed results. Finally, some conclusions are reported in Section 8.

2. Preliminaries

The construction of the Runge–Kutta methods relies heavily on the well-known simplifying assumptions [9]

$$B(p) : \sum_{i=1}^{s} b_i c_i^{n-1} = \frac{1}{q}, q = 1, \ldots, p;$$

$$C(\eta) : \sum_{i=1}^{s} a_{ij} c_j^{q-1} = \frac{c_i^q}{q}, i = 1, \ldots, s, q = 1, \ldots, \eta;$$

$$D(\zeta) : \sum_{i=1}^{s} b_i c_i^{q-1} a_{ij} = \frac{b_j}{q}(1 - c_j^q), j = 1, \ldots, s, q = 1, \ldots, \zeta,$$

based on the following fundamental theorem of Butcher.

Theorem 2.1 [7]. *If the coefficients b_i, c_i, a_{ij} of a Runge–Kutta method satisfy $B(p)$, $C(\eta)$, $D(\zeta)$ with $p \le \eta + \zeta + 1$ and $p \le 2\eta + 2$, then the method is of order p.*

Now, we recall the *W*-transformation, which was proposed by Hairer and Wanner to construct implicit Runge–Kutta methods (see, [9]).