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Euclidean norm estimates of the inverse of some special block matrices

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A B S T R A C T

In this paper we start from the known lower bounds for minimal singular value of the matrices possessing certain kind of the diagonal dominance property, and derive Euclidean norm estimates of the inverses of several new subclasses of the block H-matrices. The motivation comes from applications where the matrix in question has distinguished block structure, which can be exploited to obtain useful information. An example arising from ecological modeling illustrates the benefits of the presented approach.

1. Introduction

Good estimations of the norm of the matrix inverse can be useful in many applications, especially in error and convergence analysis of matrix splitting iterative methods applied to linear systems, arising from discretization of partial differential equations. When the system matrix has a block form, like, for example, in the case of parallel block-wise matrix multisplitting and two-stage multi splitting iteration methods (see [1–3]), it is very useful to use this particular structure, in order to obtain possibly better conditioning estimates, comparing to the point-wise case. This was the main motivation for developing such infinity norm estimates for some special subclasses of block H-matrices in [4]. Here, we consider different classes of nonsingular matrices and present Euclidean norm estimates of their inverse.

The paper is organized as follows: Section 2 presents block generalizations defined in [11,13], using the Euclidean norm. Section 3 presents connection between point-wise and block case, Section 4 is an overview of known estimations for the minimal singular value in the point-wise case, Section 5 contains the main results for block case, and, finally, Section 6 illustrates usefulness and efficiency of new Euclidean norm estimates, especially on the practical example from the ecological modeling of soil food webs.

Throughout this paper, we denote by $N := \{1, 2, \dots, n\}$ set of indices. For a matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$ we define

$$r_i(A) := \sum_{j \in N \setminus \{i\}} |a_{ij}|, \quad i \in N, \quad \text{and} \quad c_i(A) := r_i(A^T).$$

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By $\pi = \{p_j\}_{j=0}^{\ell}$ we always denote a partition of the index set N, if nonnegative numbers $p_j,\ j = 1, 2, \dots, \ell,$ satisfy the following condition

$$p_0 := 0 < p_1 < p_2 < \cdots < p_\ell := n.$$

Then, by this partition, an $n \times n$ matrix A is partitioned into $\ell \times \ell$ blocks

$$A = \left[\begin{array}{c|c|c|c} A_{11} & A_{12} & \cdots & A_{1\ell} \\ \hline A_{21} & A_{22} & \cdots & A_{2\ell} \\ \hline \vdots & \vdots & & \vdots \\ \hline A_{\ell 1} & A_{\ell 2} & \cdots & A_{\ell\ell} \end{array}\right] = [A_{ij}]_{\ell\times\ell}. \tag{1}$$

It is well known that the Euclidean matrix norm is defined by

$$\|A\|_2 = \sqrt{\rho(A^H A)} = \sqrt{\rho(AA^H)}.$$

Let us recall some of well-known (point-wise) classes of matrices. One of the widest one is the class of nonsingular H-matrices, defined as matrices $A = [a_{ij}] \in \mathbb{C}^{n,n}$, for which the comparison matrix $\mathcal{M}(A) = [\alpha_{ij}]$ defined by

$$\mathcal{M}(A) = [\alpha_{ij}] \in \mathbb{C}^{n,n}, \qquad \alpha_{ij} = \begin{cases} |a_{ii}|, & i = j \\ -|a_{ij}|, & i \neq j \end{cases}$$

is an M-matrix, i.e., $\mathcal{M}(A)^{-1} \geq 0$.

The most important subclass of nonsingular H-matrices is the class of strictly diagonally dominant (*SDD*) matrices, defined as matrices $A = [a_{ij}] \in \mathbb{C}^{n,n}$, for which

$$|a_{ii}| > r_i(A) \ \text{ for all } \ i \in N.$$

2. Two possibilities for block generalizations

For any block matrix $A = [A_{ij}]_{\ell\times\ell}$ of the form (1), we will construct two, generally different, comparison $\ell \times \ell$ real matrices. We will denote them by $\langle A\rangle^{\pi}$ and $\rangle A\langle^{\pi}$, in order to emphasize that they depend on the partition π. First comparison matrix $\langle A\rangle^{\pi}$ is constructed in the same manner as it was done in [13], while the second one $\rangle A\langle^{\pi}$ is constructed as it was done in [11].

More precisely, a partition π of the set of indices N can be treated as a partition π of $\mathbb{C}^n$, i.e., a finite collection $\{W_i\}_{i=1}^{\ell}$ of pairwise disjoint linear subspaces, each having dimension at least 1, whose direct sum is $\mathbb{C}^n$:

$$\mathbb{C}^n = W_1 \oplus W_2 \oplus \cdots \oplus W_\ell.$$

It is assumed that $W_j = span\{e^k : p_{j-1} + 1 \leq k \leq p_j\},\ j \in L := \{1, 2, \dots, \ell\}$. (Vectors $e^1, e^2, \dots, e^n$ denote the standard column basis vectors in $\mathbb{C}^n$.)

As usual, denote

$$\|A_{ij}\|_2 := \sup_{x\in W_j\setminus\{0\}} \frac{\|A_{ij}x\|_2}{\|x\|_2} = \sup_{\|x\|_2=1} \|A_{ij}x\|_2, \tag{2}$$

and

$$(\|A_{ii}^{-1}\|_2)^{-1} := \inf_{x\in W_i\setminus\{0\}} \frac{\|A_{ii}x\|_2}{\|x\|_2}, \quad i \in L. \tag{3}$$

If A_{ii} is nonsingular, relation (3) corresponds to the definition of Euclidean norm, while if A_{ii} is singular, then $\inf_{x\in W_i\setminus\{0\}} \frac{\|A_{ii}x\|_2}{\|x\|_2} = 0$.

With this notations, define:

- *the first comparison matrix* $\langle A\rangle^{\pi} = [\mu_{ij}]$ with

$$\mu_{ij} := \begin{cases} (\|A_{ii}^{-1}\|_2)^{-1}, & i = j \text{ and } A_{ii} \text{ is nonsingular} \\ 0 & i = j \text{ and } A_{ii} \text{ is singular} \\ -\|A_{ij}\|_2, & i \neq j \end{cases}$$

the second comparison matrix $\rangle A\langle^{\pi} = [m_{ij}]$ with

$$m_{ij} := \begin{cases} 1, & i = j \text{ and } A_{ii} \text{ is nonsingular} \\ -\|A_{ii}^{-1}A_{ij}\|_2, & i \neq j \text{ and } A_{ii} \text{ is nonsingular} \\ 0 & \text{otherwise} \end{cases}$$