Applied Mathematics and Computation 284 (2016) 37–46

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

# Reverse order laws for the generalized strong Drazin inverses

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ARTICLE INFO

*MSC:*
46H05
16B99
15A09

*Keywords:*
Strong Drazin inverse
Reverse order law
Rings
Banach algebras

ABSTRACT

Various kinds of the reverse order laws for the strong Drazin inverse are characterized in a ring. Then we define the generalized strong Drazin inverse in a Banach algebra and present similar results related to reverse order laws for this new inverse.



## 1. Introduction

Let $\mathcal{R}$ be an associative ring with the unit 1. The set of all nilpotent elements of $\mathcal{R}$ will be denoted by $\mathcal{R}^{nil}$.

An element $a \in \mathcal{R}$ is strongly Drazin invertible (or $s$–Drazin invertible) if there exists $x \in \mathcal{R}$ such that

$$(2)\ xax = x, \quad (5)\ ax = xa, \quad (7)\ a - ax \in \mathcal{R}^{nil};$$

$x$ is a strong Drazin inverse of $a$. If the strong Drazin inverse of $a$ exists, it is unique and denoted by $a'$ [21]. The least positive integer $n$ such that $(a - aa')^n = 0$ is called the index of $a$, denoted by $ind(a)$. The strong Drazin inverse $a'$ double commutes with $a$, that is, $ay = ya$ implies $a'y = ya'$ [21]. Denote by $\mathcal{R}^{sD}$ the set of all strongly Drazin invertible elements of $\mathcal{R}$.

If $\delta \subseteq \{2, 5, 7\}$ and $b$ satisfies the equations $(i)$ for all $i \in \delta$, then $b$ is an $\delta$-inverse of $a$. The set of all $\delta$-inverse of $a$ is denoted by $a\{\delta\}$. Observe that $a\{2, 5, 7\} = \{a'\}$.

**Lemma 1.1** ([21, Proposition 1.7]). *Let $a \in \mathcal{R}^{sD}$. Then the s-Drazin inverse $a'$ is also s-Drazin invertible. In this case, $(a')' = a^2a'$ and $ind(a') \leq ind(a)$.*

**Lemma 1.2.** *Let $a, b \in \mathcal{R}$ be nilpotent and $ab = ba$. Then $a + b$ is also nilpotent.*

If we replace the condition $a - ax \in \mathcal{R}^{nil}$ in the definition of the strong Drazin invertible element with $a(1 - ax) \in \mathcal{R}^{nil}$, then the element $a$ is Drazin invertible [4]. Let $a \in \mathcal{R}^{sD}$. Then $a(1 - aa') = (a - aa')(1 - aa') \in \mathcal{R}^{nil}$. So, every strong Drazin invertible element is Drazin invertible, but the converse is not true. For example, 2 is (Drazin) invertible in complex number field $\mathbb{C}$, but it is not $s$-Drazin invertible.

The equality $(ab)^{-1} = b^{-1}a^{-1}$ which holds for invertible elements $a, b \in \mathcal{R}$, is called the reverse order law. If we replace the inverse with some type of generalized inverses in the previous equality, we obtain a class of interesting problems. Since the reverse order law for the generalized inverse is an useful computational tool in applications and it is significant from the

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http://dx.doi.org/10.1016/j.amc.2016.02.051
0096-3003/

theoretical point of view, many papers characterized the reverse order laws in the setting of matrices, operators, elements of $C^*$-algebras or rings [3,5,10,13,15,19,20].

The reverse order laws for the (generalized) Drazin inverse are investigated in [16]. In [1,2,14,17,18], it can be found different characterizations of the reverse order laws for the group inverse, which is a special case of the Drazin inverse. Some recent applications of Drazin inverses can be found in [6–9,12].

Considering the reverse order law for the strong Drazin inverse, Wang [21] proved that: if $a, b \in \mathcal{R}^{sD}$ and $ab = ba$, then $ab \in \mathcal{R}^{sD}$ and $(ab)' = b'a'$.

In this paper, we study various types of the reverse order laws for the strong Drazin inverses, obtaining the equivalences between some of them. Precisely, in Section 2, the reverse order laws for the strong Drazin inverse in a ring are investigated in the cases that the product $ab$ commutes with $a$ or (and) $b$, the element $a'ab$ commutes with $b$ and the element $abb'$ commutes with $a$. In Section 3, we introduce the generalized strong Drazin inverse in a Banach algebra and present similar results for this inverse.

## 2. Reverse order laws for the strong Drazin inverse

Under certain assumptions, various types of the reverse order laws for the strong Drazin inverse in a ring will be investigated in this section.

When $ab$ commutes with $a$, necessary and sufficient conditions for the reverse order laws $(ab)' = (a'ab)'a'$ and $(a'ab)' = (ab)'a$ are considered in the next theorem.

**Theorem 2.1.** *Let $b \in \mathcal{R}$ and $a \in \mathcal{R}^{sD}$. If the element $ab$ commutes with $a$, then the following statements are equivalent:*

(i) $ab, a'ab \in \mathcal{R}^{sD}$ *and* $(ab)' = (a'ab)'a'$,
(ii) $a'ab \in \mathcal{R}^{sD}$ *and* $(a'ab)'a' \in (ab)\{7\}$,
(iii) $ab, a'ab \in \mathcal{R}^{sD}$ *and* $(a'ab)' = (ab)'a$,
(iv) $ab \in \mathcal{R}^{sD}$ *and* $(ab)'a \in (a'ab)\{2\}$,
(v) $ab \in \mathcal{R}^{sD}$ *and* $(ab)'a(1 - a'a) = 0$.

**Proof.** (i) $\Rightarrow$ (iii): The assumption $(ab)' = (a'ab)'a'$ gives $(ab)'aa' = (ab)'$. Now, we can check that $(ab)'a \in (a'ab)\{2\}$. The element $ab$ commutes with $a$ and the strong Drazin inverse $a'$ double commutes with $a$, so $ab$ commutes with $a'$. Therefore, $a'$ commutes with $(ab)'$ and $a'ab$ commutes with $(ab)'$ and $a$, that is $(ab)'a \in (a'ab)\{5\}$. Observe that $(ab)'a \in (a'ab)\{7\}$, because $a'a$ commutes with $a'ab$ and $(a'ab)'$ and

$$\begin{aligned} a'ab - a'ab(ab)'a &= a'ab - a'ab(a'ab)'a'a = a'ab - a'aa'ab(a'ab)' \\ &= a'ab - a'ab(a'ab)' \in \mathcal{R}^{nil}. \end{aligned}$$

Hence, $(a'ab)' = (ab)'a$.

(iii) $\Rightarrow$ (ii): By the hypothesis $(a'ab)' = (ab)'a$ and because $a$ and $a'$ commute with $ab$ and $(a'ab)'$, we deduce that

$$\begin{aligned} ab - ab(a'ab)'a' &= ab - a'ab(a'ab)' = ab - (a'ab)'a'ab \\ &= ab - a'a(a'ab)'b = ab - a'aa'ab[(a'ab)']^2 b \\ &= ab - (a'ab)'b = ab - (ab)'ab \in \mathcal{R}^{nil}. \end{aligned}$$

(ii) $\Rightarrow$ (i): First, note that $(a'ab)'a' \in (ab)\{2\}$. Since $a'ab$ commutes with $ab$, then $(a'ab)'$ commutes with $ab$ and $(a'ab)'a' \in (ab)\{5\}$. Thus, $ab \in \mathcal{R}^{sD}$ and $(ab)' = (a'ab)'a'$.

(iii) $\Rightarrow$ (iv): It is evident.

(iv) $\Rightarrow$ (v): From $(ab)'a \in (a'ab)\{2\}$, we get that (v) holds:

$$(ab)'a = (ab)'aa'ab(ab)'a = (ab)'ab(ab)'aa'a = (ab)'aa'a.$$

(v) $\Rightarrow$ (iii): Applying $(ab)'aa'a = (ab)'a$, we obtain that $(ab)'a \in (a'ab)\{2\}$. As in the part (i) $\Rightarrow$ (iii), we conclude that $(ab)'a \in (a'ab)\{5\}$. Recall that $ab - ab(ab)' \in \mathcal{R}^{nil}$. Since $aa'$ commutes with $ab - ab(ab)'$, we deduce that $aa'(ab - ab(ab)') \in \mathcal{R}^{nil}$. By Lemma 1.1, $a' - a'a = a' - a'(a')' \in \mathcal{R}^{nil}$. Because $ab$ commutes with $a' - a'a$, we have $(a' - aa')ab \in \mathcal{R}^{nil}$. Using Lemma 1.2, observe that

$$\begin{aligned} a'ab - a'ab(ab)'a &= a'ab - aa'ab(ab)' \\ &= a'ab - aa'ab + aa'ab - aa'ab(ab)' \\ &= (a' - aa')ab + aa'(ab - ab(ab)') \in \mathcal{R}^{nil}, \end{aligned}$$

i.e. the statement (iii) is satisfied. □

If we define reverse multiplication in a ring $\mathcal{R}$ by $a \circ b = ba$, we obtain the opposite ring $(\mathcal{R}, \circ)$. Applying Theorem 2.1 to the opposite ring $(\mathcal{R}, \circ)$, we get the dual statement concerning $(ab)' = b'(abb')'$ and $(abb')' = b(ab)'$.

**Corollary 2.1.** *Let $a \in \mathcal{R}$ and $b \in \mathcal{R}^{sD}$. If the element $ab$ commutes with $b$, then the following statements are equivalent:*