Applied Mathematics and Computation 284 (2016) 79–88

Contents lists available at ScienceDirect

Applied Mathematics and Computation

journal homepage: www.elsevier.com/locate/amc

On the numerical treatment and analysis of Benjamin–Bona–Mahony–Burgers equation

M. Zarebnia*, R. Parvaz

Department of Mathematics, University of Mohaghegh Ardabili, 56199-11367 Ardabil, Iran

ARTICLE INFO

Keywords:
Benjamin–Bona–Mahony–Burgers
B-spline
Collocation
Stability
Convergence

ABSTRACT

In this paper, the solution of Benjamin–Bona–Mahony–Burgers (BBMB) equation is approximated by using collocation method. Stability of the method is discussed. Also the convergence of the method is proved. The method is used on some examples, and the numerical results have been obtained and compared with the exact solutions.

1. Introduction

In the present work, we will consider the BBMB equation. This equation given as follows:

$$u_t - u_{xxt} - \alpha u_{xx} + \beta u_x + uu_x = 0, \qquad x \in [a, b], \quad t \in [0, T], \tag{1}$$

$$u(x, 0) = f(x), \qquad x \in [a, b], \tag{2}$$

$$u(a, t) = u(b, t) = 0, \tag{3}$$

where α and β are positive constants. BBMB equations are used in many branches of science and engineering, for example see [1]. When $\alpha = 0$, Eq. (1) is called the Benjamin–Bona–Mahony (BBM) equation. In recent years, various type of methods have been used to estimate the solution of the BBMB equation [2–6].

The layout of the paper is as follows: in Section 2, cubic B-spline collocation method is explained. In Section 3, we develop an algorithm for the numerical solution of the BBMB equation. Section 4, is devoted to stability analysis of the method. We derive convergence of the B-spline collocation method in Section 5. In Section 6, examples are presented. A summary is given at the end of the paper in Section 7.

2. B-spline collocation method

The interval $[a, b]$ is partitioned into a mesh of uniform length $h = (b - a)/N$ by the knots $x_i = a + ih, i = -3, \ldots, N+3$ such that $a = x_0 < x_1 < x_2 < \cdots < x_{N-1} < x_N = b$. Our numerical treatment for BBMB equation using the collocation method with cubic B-spline is to find an approximate solution $U(x, t)$ to the exact solution $u(x, t)$ in the form

$$U(x, t) = \sum_{i=-1}^{N+1} c_i(t) B_i(x), \tag{4}$$

* Corresponding author. Tel.: +98 4533514702; fax: +98 4533514701.
E-mail addresses: zarebnia@uma.ac.ir (M. Zarebnia), rparvaz@uma.ac.ir (R. Parvaz).

http://dx.doi.org/10.1016/j.amc.2016.02.037
0096-3003/

where $c_i(t)$ are time-dependent quantities to be determined from the boundary conditions and collocation form of the differential equations. Also $B_i(x)$ are the cubic B-spline basis functions at knots, given by Stoer and Bulirsch [7], Kincad and Cheny [8] and Rubin and Graves [9]

$$B_i(x) = \frac{1}{6h^3}\begin{cases}(x-x_{i-2})^3, & x\in[x_{i-2},x_{i-1}),\\ h^3+3h^2(x-x_{i-1})+3h(x-x_{i-1})^2-3(x-x_{i-1})^3, & x\in[x_{i-1},x_i),\\ h^3+3h^2(x_{i+1}-x)+3h(x_{i+1}-x)^2-3(x_{i+1}-x)^3, & x\in[x_i,x_{i+1}),\\ (x_{i+2}-x)^3, & x\in[x_{i+1},x_{i+2}),\\ 0, & \text{otherwise.}\end{cases}$$

The values of U and its space derivatives at the knots x_i can be obtained as

$$U_i = \frac{1}{6}(c_{i-1}+4c_i+c_{i+1}), \tag{5}$$

$$U_i' = \frac{1}{2h}(-c_{i-1}+c_{i+1}), \tag{6}$$

$$U_i'' = \frac{1}{h^2}(c_{i-1}-2c_i+c_{i+1}). \tag{7}$$

3. Construction of the method

Now we use the following finite difference approximation to discretize the time variable

$$\frac{u^{n+1}-u^n}{\Delta t}-\frac{u_{xx}^{n+1}-u_{xx}^n}{\Delta t}-\alpha\frac{u_{xx}^{n+1}+u_{xx}^n}{2}+\beta\frac{u_x^{n+1}+u_x^n}{2}+\frac{(uu_x)^{n+1}+(uu_x)^n}{2}=0, \tag{8}$$

where Δt is the time step. In fact we use forward Euler scheme for Eq. (1). The nonlinear term in Eq. (8) can be approximated by using the following formula:

$$(uu_x)^{n+1}=(uu_x)^n+\Delta t(uu_x)_t^n+\Delta t^2(uu_x)_{tt}^n+O(\Delta t^2)=(uu_x)^n+\Delta t\left(\frac{u^{n+1}-u^n}{\Delta t}u_x^n+\frac{u_x^{n+1}-u_x^n}{\Delta t}u^n\right)+\Delta t^2\left(\frac{u^{n+1}-2u^n+u^{n-1}}{\Delta t^2}u_x^n+2\frac{u^{n+1}-u^n}{\Delta t}\frac{u_x^{n+1}-u_x^n}{\Delta t}+\frac{u_x^{n+1}-2u_x^n+u_x^{n-1}}{\Delta t^2}u^n\right)+O(\Delta t^2),$$

thus we have

$$(uu_x)^{n+1}=3(uu_x)^n-u^{n-1}u_x^n-u^nu_x^{n-1}+O(\Delta t^2). \tag{9}$$

Putting values from Eq. (9) in Eq. (8) and simplifying we get

$$u^{n+1}+su_x^{n+1}+qu_{xx}^{n+1}=\Psi^n, \tag{10}$$

where

$$\Psi^n=u^n-\frac{\beta\Delta t}{2}u_x^n+\left(-1+\frac{\alpha\Delta t}{2}\right)u_{xx}^n-\frac{\Delta t}{2}(4u^nu_x^n-u^{n-1}u_x^n-u^nu_x^{n-1}), \tag{11}$$

and

$$s=\frac{\beta\Delta t}{2},\ q=-1-\frac{\alpha\Delta t}{2}.$$

Substituting the approximate solution U for u and putting the values of the nodal values U, its derivatives using Eqs. (5)–(7) at the knots in Eq. (10), we obtain the following difference equation with the variables c_i, $\quad i=-1,\dots,N+1$,

$$\acute{a}c_{i-1}^{n+1}+\acute{b}c_i^{n+1}+\acute{c}c_{i+1}^{n+1}=h^2\Psi^n(x_i),\quad i=0,1,\dots,N, \tag{12}$$

where

$$\acute{a}=\frac{h^2}{6}-\frac{sh}{2}+q,\quad \acute{b}=\frac{4h^2}{6}-2q,\quad \acute{c}=\frac{h^2}{6}+\frac{sh}{2}+q.$$

The system (12) consists of $N+1$ linear equations and $N+3$ unknowns $\{c_{-1},c_0,\dots,c_N,c_{N+1}\}$. To obtain a unique solution for $\{c_{-1},\dots,c_{N+1}\}$, we must use the boundary conditions. From the boundary conditions and Eq. (5), we can write

$$c_{-1}^{n+1}=-4c_0^{n+1}-c_1^{n+1}, \tag{13}$$

$$c_{N+1}^{n+1}=-4c_N^{n+1}-c_{N-1}^{n+1}. \tag{14}$$

Also putting $i=0,N$ in Eq. (12) and using, Eqs. (13) and (14) we get the results as:

$$(-4\acute{a}+\acute{b})c_0^{n+1}+(-\acute{a}+\acute{c})c_1^{n+1}=h^2\Psi^n(x_0), \tag{15}$$

$$(\acute{a}-\acute{c})c_{N-1}^{n+1}+(\acute{b}-4\acute{c})c_N^{n+1}=h^2\Psi^n(x_N). \tag{16}$$