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Taylor series approach for function approximation using 'estimated' higher derivatives

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ABSTRACT

The paper proposes a new approach for function approximation by 'estimating' the second derivative, using Taylor series, where the samples of the function and its first derivatives at the sample points are known. Without such 'estimation', these initial data, can approximate the function traditionally using the first order Taylor approximation, but with more error. If we desire to improve the approximation via second order Taylor series, then we can estimate the 'pseudo' second derivatives of the function in three different ways. All these three ways are investigated in this paper. The pseudo second derivatives help in computing many more sample points of the function within each sampling interval. Thus, the approach acts like a mathematical 'magnifying glass'. Two examples are treated to compare the efficiencies of the methods. Also, a qualitative study for upper bound of error of the approximations is studied in detail.

1. Introduction

The well known Taylor series [1,2], introduced by the English mathematician Brook Taylor in 1715, has been a useful mathematical tool through centuries, and is applied in many areas of mathematics still today.

Taylor series has many varieties of applications. Significant work has been done in the area of function approximations and many kinds of differential equations. The operational matrix technique [3,4], quite popular with piecewise constant orthogonal functions like Haar functions, Walsh functions, block pulse functions [3,5] etc., has also been derived using the Taylor series [6]. In many cases, the Taylor series approach comes up with fast algorithms suitable for computer applications.

In the area of mathematics, many efficient algorithms like the Taylor based approach have been developed in recent times. In 2004, Dehghan used finite difference procedures for solving one-dimensional advection–diffusion equation [7] and also for solving the problem of modeling and design of certain optoelectronic devices [8].

The year 2006 saw the work of Mohebbi and Dehghan [9] who used a multigrid method having grid-independent convergence and solved a linear system of equations within a small computation time. They used fourth-order finite difference approximations to solve test problems producing highly accurate solutions. In 2010, they proposed [10] a high-order accurate method for solving the one-dimensional heat and advection–diffusion equations employing a compact finite difference approximation of fourth-order for discretizing spatial derivatives and the cubic C1-spline collocation method for the resulting linear system of ordinary differential equations.

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Very recently Moghaderi and Dehghan [11] came up with an efficient multigrid method to obtain the numerical solution of the one-dimensional nonlinear sine-Gordon equation using a compact finite difference scheme of fourth-order for discretizing the spatial derivative and the standard second-order central finite difference method for the time derivative.

In the area of control theory also, quality algorithms are sought for by modern researchers. We have seen the use of algorithms based upon Taylor series for solving state equations [12], time delay systems [13,14], multi-delay systems [15], integral equations [16], optimal control problems [17,18] and many others. Thus, the Taylor series is keeping researchers quite busy even today.

This paper proposes an algorithm to approximate a time function based upon Taylor series, where the order of approximation is improved by using an 'estimated' one step higher derivative from the basic available data. These are the samples of the function, with sampling period h, and its first derivatives at sampling instants. The presented method 'estimates' the second derivative numerically and approximates the function in an improved manner.

If we work only with the samples of a function, traditionally, by computing the first derivative numerically within each sampling interval, the function may be represented in a piecewise linear manner. This computed derivative is, in fact, the *average slope* of the function in *that* particular interval. This process ensures, the approximated version of the said function passes through *all the samples* of the original function itself.

However, the use of the second derivative results in a better approximation. For 'estimation' of second derivatives, three alternatives have been suggested in this work and their quality of estimation has been studied with examples. The theory of the best estimator has been extended to the pth order derivative as well in a very general manner.

The numerical examples are treated exhaustively to reveal all aspects of the proposed theory. The upper bound of error for all estimated functions has been presented to prove accuracy of function approximation by three modifications and also error for a particular example is estimated to study for choosing the best option.

2. Function approximation via Taylor series

Consider a square integrable function $f(t)$ defined over an interval $t \in [0, T)$. Let $T=mh$, where, h is the width of the equidistant sub-intervals and m=0, 1, 2, …, N, N being a large number. The function $f(t)$ is approximated around the point μ_i via Taylor series as

$$f(t) \approx f(\mu_i) + \dot{f}(\mu_i)(t-\mu_i) + \ddot{f}(\mu_i)\frac{(t-\mu_i)^2}{2!} + \dddot{f}(\mu_i)\frac{(t-\mu_i)^3}{3!} + \cdots$$

Assuming $\mu_i = kh$,

$$f(t) \approx f(kh) + \dot{f}(kh)(t-kh) + \ddot{f}(kh)\frac{(t-kh)^2}{2!} + \dddot{f}(kh)\frac{(t-kh)^3}{3!} + \cdots = \bar{f}(t) \quad \text{(say)} \tag{1}$$

Considering the approximation within one sub-interval $[kh, (k+1)h]$, Eq. (1) becomes,

$$f[(k+1)h] \approx f(kh) + h\dot{f}(kh) + \frac{h^2}{2!}\ddot{f}(kh) + \frac{h^3}{3!}\dddot{f}(kh) + \cdots \tag{2}$$

If h is small enough, we can write

$$f[(k+1)h] \approx \bar{f}[(k+1)h].$$

This can also reasonably be achieved using second order Taylor approximation. Hence, Eq. (2) reduces to

$$\bar{f}[(k+1)h] = f(kh) + h\dot{f}(kh) + \frac{h^2}{2!}\ddot{f}(kh) \tag{3}$$

In case only the samples of the function are available, we can join these either using the zero-th order Taylor series, like the zero order hold (ZOH), or approximate the function in a piece wise linear manner by joining the consecutive samples with a straight line. If, in addition to the samples, the first order derivative of the function at each sample point is known, the approximation, obviously, may be improved.

A function $f(t)$ is shown in Fig. 1, where $f(kh)$ and $f[(k+1)h]$ are its two samples in the interval $t \in [kh, (k+1)h]$. We intend to approximate this function by joining the samples $f(kh)$ and $f[(k+1)h]$ using a straight line and also, via first order Taylor series.

Let the piece wise linear approximation of $f(t)$ be called $\hat{f}(t)$.

For first order Taylor series approximation, we assume that, in addition to the sample points, the first order derivative $\dot{f}(kh)$ of $f(t)$ at kh is known. This first order approximation yields the value $\bar{f}[(k+1)h]$ at the point $(k+1)h$. Joining the line between $f(kh)$ and $\bar{f}[(k+1)h]$ produces the first order Taylor approximation of $f(t)$, which we call $\bar{f}(t)$. In Fig. 1, the approximations are extended to the next interval for elaboration.

If the width of the interval h tends to zero, for non-oscillatory functions, the point $\bar{f}[(k+1)h]$ approaches $f[(k+1)h]$ in Fig. 1 and the two functions $\bar{f}(t)$ and $\hat{f}(t)$ coincide.

For a predefined h, both the above approximation incurs errors. That is, for the piece wise linear approximation all information about the function between two consecutive sample points is lost. And for Taylor approximation, apart from